6

GEOMETRY

There is geometry in the humming of the strings, there is music in the spacing of spheres - Pythagoras

- Introduction
- Basic Proportionality Theorem
- Angle Bisector Theorem
- Similar Triangles
- Tangent chord theorem
- Pythagoras theorem

EUCLID
(300 BC)
Greece

Euclid's 'Elements' is one of the most influential works in the history of mathematics, serving as the main text book for teaching mathematics especially geometry.

Euclid's algorithm is an efficient method for computing the greatest common divisor.

6.1 Introduction

Geometry is a branch of mathematics that deals with the properties of various geometrical figures. The geometry which treats the properties and characteristics of various geometrical shapes with axioms or theorems, without the help of accurate measurements is known as theoretical geometry. The study of geometry improves one's power to think logically.

Euclid, who lived around 300 BC is considered to be the father of geometry. Euclid initiated a new way of thinking in the study of geometrical results by deductive reasoning based on previously proved results and some self evident specific assumptions called axioms or postulates.

Geometry holds a great deal of importance in fields such as engineering and architecture. For example, many bridges that play an important role in our lives make use of congruent and similar triangles. These triangles help to construct the bridge more stable and enables the bridge to withstand great amounts of stress and strain. In the construction of buildings, geometry can play two roles; one in making the structure more stable and the other in enhancing the beauty. Elegant use of geometric shapes can turn buildings and other structures such as the Taj Mahal into great landmarks admired by all. Geometric proofs play a vital role in the expansion and understanding of many branches of mathematics.

The basic proportionality theorem is attributed to the famous Greek mathematician Thales. This theorem is also called Thales theorem.

To understand the basic proportionality theorem, let us perform the following activity.

Activity

Draw any angle XAY and mark points (say five points) P_1, P_2, D, P_3 and B on arm AX such that $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$ unit (say).

Through B draw any line intersecting arm AY at C. Again through D draw a line parallel to BC to intersect AC at E.

Now $\quad AD = AP_1 + P_1P_2 + P_2D = 3$ units

and $\quad DB = DP_3 + P_3B = 2$ units

$\therefore \quad \dfrac{AD}{DB} = \dfrac{3}{2}$

Measure AE and EC.

We observe that $\dfrac{AE}{EC} = \dfrac{3}{2}$

Thus, in ΔABC if $DE \parallel BC$, then $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

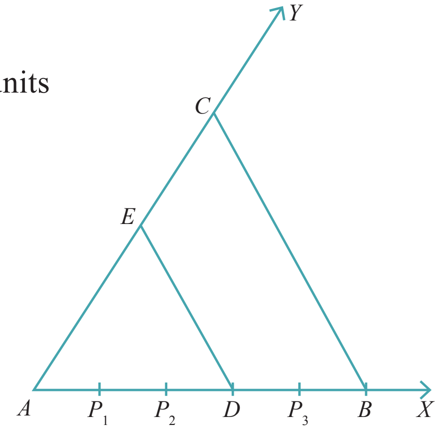


Fig. 6.1

We prove this result as a theorem known as Basic Proportionality Theorem or Thales Theorem as follows:

6.2 Basic proportionality and Angle Bisector theorems

Theorem 6.1 **Basic Proportionality theorem or Thales Theorem**

If a straight line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.

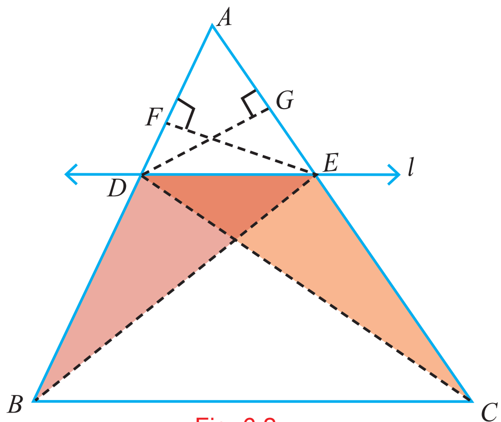


Fig. 6.2

Given: In a triangle ABC, a straight line l parallel to BC, intersects AB at D and AC at E.

To prove: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Construction: Join BE, CD.

Draw $EF \perp AB$ and $DG \perp CA$.

Proof

Since, $EF \perp AB$, EF is the height of triangles ADE and DBE.

Area $(\Delta ADE) = \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} AD \times EF$ and

Area $(\Delta DBE) = \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} DB \times EF$

$$\therefore \quad \frac{\text{area}(\Delta ADE)}{\text{area}(\Delta DBE)} = \frac{\frac{1}{2}AD \times EF}{\frac{1}{2}DB \times EF} = \frac{AD}{DB} \qquad (1)$$

Similarly, we get

$$\frac{\text{area}(\Delta ADE)}{\text{area}(\Delta DCE)} = \frac{\frac{1}{2} \times AE \times DG}{\frac{1}{2} \times EC \times DG} = \frac{AE}{EC} \qquad (2)$$

But, ΔDBE and ΔDCE are on the same base DE and between the same parallel straight lines BC and DE.

$$\therefore \quad \text{area }(\Delta DBE) = \text{area }(\Delta DCE) \qquad (3)$$

Form (1), (2) and (3), we obtain $\frac{AD}{DB} = \frac{AE}{EC}$. Hence the theorem.

Corollary

If in a ΔABC, a straight line DE parallel to $BC,$ intersects AB at D and AC at $E,$ then

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

Proof

(i) From Thales theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\Longrightarrow \frac{DB}{AD} = \frac{EC}{AE}$$

$$\Longrightarrow 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\Longrightarrow \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$$

Thus, $\frac{AB}{AD} = \frac{AC}{AE}$

(ii) Similarly, we can prove

$$\frac{AB}{DB} = \frac{AC}{EC}$$

Do you know?

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

This is called componendo rule.

Here, $\frac{DB}{AD} = \frac{EC}{AE}$

$\Rightarrow \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$

by componendo rule.

Is the converse of this theorem also true? To examine this let us perform the following activity.

Activity

Draw an angle $\angle XAY$ and on the ray AX, mark points P_1, P_2, P_3, P_4 and B such that

$AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4B = 1$ unit (say).

Similarly, on ray AY, mark points Q_1, Q_2, Q_3, Q_4 and C such that

$AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4C = 2$ units (say).

Now join P_1Q_1 and BC.

Then $\frac{AP_1}{P_1B} = \frac{1}{4}$ and $\frac{AQ_1}{Q_1C} = \frac{2}{8} = \frac{1}{4}$

Thus, $\frac{AP_1}{P_1B} = \frac{AQ_1}{Q_1C}$

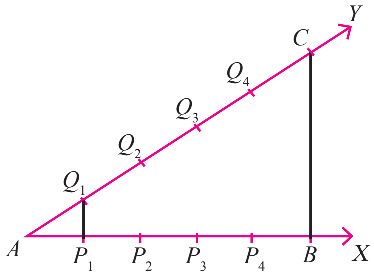


Fig. 6.3

We observe that the lines P_1Q_1 and BC are parallel to each other. i.e., $P_1Q_1 \parallel BC$ (1)

Similarly, by joining P_2Q_2, P_3Q_3 and P_4Q_4 we see that

$\frac{AP_2}{P_2B} = \frac{AQ_2}{Q_2C} = \frac{2}{3}$ and $P_2Q_2 \parallel BC$ (2)

$\frac{AP_3}{P_3B} = \frac{AQ_3}{Q_3C} = \frac{3}{2}$ and $P_3Q_3 \parallel BC$ (3)

$\frac{AP_4}{P_4B} = \frac{AQ_4}{Q_4C} = \frac{4}{1}$ and $P_4Q_4 \parallel BC$ (4)

From (1), (2), (3) and (4) we observe that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

In this direction, let us state and prove a theorem which is the converse of Thales theorem.

Theorem 6.2

Converse of Basic Proportionality Theorem (Converse of Thales Theorem)

If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

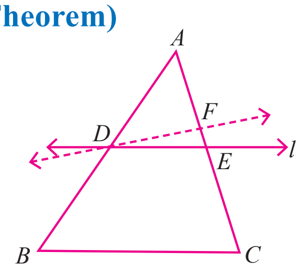


Fig. 6.4

Given: A line l intersects the sides AB and AC of ΔABC respectively at D and E such that $\frac{AD}{DB} = \frac{AE}{EC}$ (1)

To prove : $DE \parallel BC$

Construction : If DE is not parallel to BC, then draw a line $DF \parallel BC$.

Proof Since $DF \parallel BC$, by Thales theorem we get,

$$\frac{AD}{DB} = \frac{AF}{FC} \quad (2)$$

From (1) and (2), we get $\frac{AF}{FC} = \frac{AE}{EC} \implies \frac{AF+FC}{FC} = \frac{AE+EC}{EC}$

$$\frac{AC}{FC} = \frac{AC}{EC} \quad \therefore\ FC = EC$$

This is possible only when F and E coincide. Thus, $DE \parallel BC$.

Theorem 6.3 Angle Bisector Theorem

The internal (external) bisector of an angle of a triangle divides the opposite side internally (externally) in the ratio of the corresponding sides containing the angle.

Case (i) (Internally)

Given : In $\triangle ABC$, AD is the internal bisector of $\angle BAC$ which meets BC at D.

To prove : $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction : Draw $CE \parallel DA$ to meet BA produced at E.

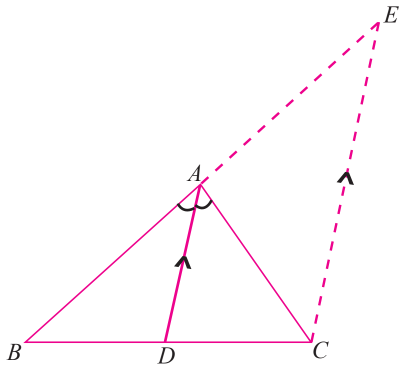


Fig. 6.5

Proof

Since $CE \parallel DA$ and AC is the transversal, we have

$\angle DAC = \angle ACE$ (alternate angles) (1)

and $\angle BAD = \angle AEC$ (corresponding angles) (2)

Since AD is the angle bisector of $\angle A$, $\angle BAD = \angle DAC$ (3)

From (1), (2) and (3), we have $\angle ACE = \angle AEC$

Thus in $\triangle ACE$, we have $AE = AC$ (sides opposite to equal angles are equal)

Now in $\triangle BCE$ we have, $CE \parallel DA$

$$\frac{BD}{DC} = \frac{BA}{AE} \quad \text{(Thales theorem)}$$

$$\Longrightarrow \quad \frac{BD}{DC} = \frac{AB}{AC} \quad (AE = AC)$$

Hence the theorem.

Case (ii) Externally (this part is not for examination)

Given: In $\triangle ABC$,

AD is the external bisector of $\angle BAC$

and intersects BC produced at D.

To prove: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction: Draw $CE \parallel DA$ meeting AB at E.

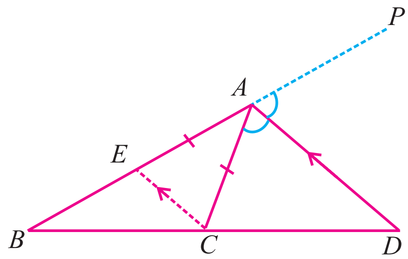


Fig. 6.6

Proof

$CE \parallel DA$ and AC is a transversal,

$\angle ECA = \angle CAD$ (alternate angles) (1)

Also $CE \parallel DA$ and BP is a transversal

$$\angle CEA = \angle DAP \qquad \text{(corresponding angles)} \qquad (2)$$

But AD is the bisector of $\angle CAP$

$$\angle CAD = \angle DAP \qquad (3)$$

From (1), (2) and (3), we have

$$\angle CEA = \angle ECA$$

Thus, in $\triangle ECA$, we have $AC = AE$ (sides opposite to equal angles are equal)

In $\triangle BDA$, we have $EC \parallel AD$

$\therefore \quad \dfrac{BD}{DC} = \dfrac{BA}{AE}$ (Thales theorem)

$\Longrightarrow \quad \dfrac{BD}{DC} = \dfrac{BA}{AC}$ $(AE = AC)$

Hence the theorem.

Theorem 6.4 Converse of Angle Bisector Theorem

If a straight line through one vertex of a triangle divides the opposite side internally (externally) in the ratio of the other two sides, then the line bisects the angle internally (externally) at the vertex.

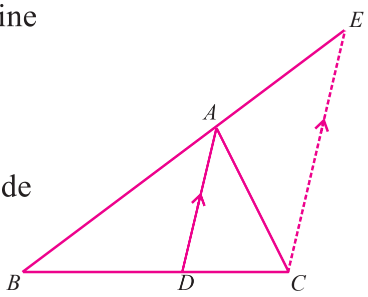


Fig. 6.7

Case (i) : (Internally)

Given : In $\triangle ABC$, the line AD divides the opposite side BC internally such that

$$\frac{BD}{DC} = \frac{AB}{AC} \qquad (1)$$

To prove : AD is the internal bisector of $\angle BAC$.

i.e., to prove $\angle BAD = \angle DAC$.

Construction :

Through C draw $CE \parallel AD$ meeting BA produced at E.

Proof Since $CE \parallel AD$, by Thales theorem, we have $\dfrac{BD}{DC} = \dfrac{BA}{AE}$ (2)

Thus, from (1) and (2) we have, $\dfrac{AB}{AE} = \dfrac{AB}{AC}$

$$\therefore \quad AE = AC$$

Now, in $\triangle ACE$, we have $\angle ACE = \angle AEC$ $(AE = AC)$ (3)

Since AC is a transversal of the parallel lines AD and CE,

we get, $\angle DAC = \angle ACE$ (alternate interior angles are equal) (4)

Also BE is a transversal of the parallel lines AD and CE.

we get $\angle BAD = \angle AEC$ (corresponding angles are equal) (5)

From (3), (4) and (5), we get

$$\angle BAD = \angle DAC$$

$\therefore$ AD is the angle bisector of $\angle BAC$.

Hence the theorem.

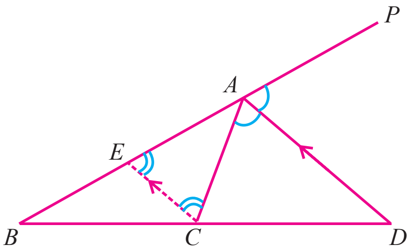


Fig. 6.8

Case (ii) Externally (this part is not for examination)

Given : In $\triangle ABC$, the line AD divides externally the opposite side BC produced at D.

such that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$ (1)

To prove : AD is the bisector of $\angle PAC$,

i.e., to prove $\angle PAD = \angle DAC$

Construction : Through C draw $CE \parallel DA$ meeting BA at E.

Proof Since $CE \parallel DA$, by Thales theorem $\dfrac{BD}{DC} = \dfrac{BA}{EA}$ (2)

From (1) and (2), we have

$$\frac{AB}{AE} = \frac{AB}{AC} \quad \therefore\ AE = AC$$

In $\triangle ACE$, we have $\angle ACE = \angle AEC$ $(AE = AC)$ (3)

Since AC is a transversal of the parallel lines AD and CE, we have

$\angle ACE = \angle DAC$ (alternate interior angles) (4)

Also, BA is a transversal of the parallel lines AD and CE,

$\angle PAD = \angle AEC$ (corresponding angles) (5)

From (3) , (4) and (5), we get

$$\angle PAD = \angle DAC$$

$\therefore$ AD is the bisector of $\angle PAC$. Thus AD is the external bisector of $\angle BAC$

Hence the theorem.

Example 6.1

In $\triangle ABC$, $DE \parallel BC$ and $\dfrac{AD}{DB} = \dfrac{2}{3}$. If $AE = 3.7$cm, find EC.

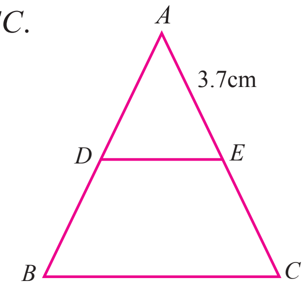


Fig. 6.9

Solution In $\triangle ABC$, $DE \parallel BC$

$\therefore$ $$\frac{AD}{DB} = \frac{AE}{EC} \quad \text{(Thales theorem)}$$

$\Longrightarrow$ $$EC = \frac{AE \times DB}{AD}$$

Thus, $$EC = \frac{3.7 \times 3}{2} = 5.55\,\text{cm}$$

Example 6.2

In $\triangle PQR$, given that S is a point on PQ such that $ST \parallel QR$ and $\dfrac{PS}{SQ} = \dfrac{3}{5}$. If $PR = 5.6$cm, then find PT.

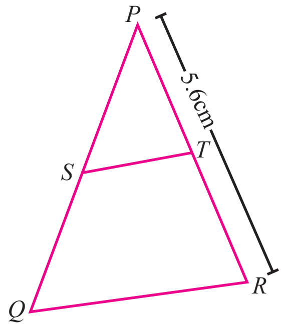


Fig. 6.10

Solution In $\triangle PQR$, we have $ST \parallel QR$ and by Thales theorem,

$$\frac{PS}{SQ} = \frac{PT}{TR} \qquad (1)$$

Let $PT = x$. Thus, $TR = PR - PT = 5.6 - x$.

From (1), we get $PT = TR\left(\dfrac{PS}{SQ}\right)$

$$x = (5.6 - x)\left(\frac{3}{5}\right)$$

$$5x = 16.8 - 3x$$

Thus, $x = \dfrac{16.8}{8} = 2.1$ That is, PT = 2.1cm.

Example 6.3

In a $\triangle ABC$, D and E are points on AB and AC respectively such that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ and $\angle ADE = \angle DEA$. Prove that $\triangle ABC$ is isosceles.

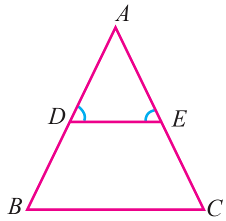


Fig. 6.11

Solution Since $\dfrac{AD}{DB} = \dfrac{AE}{EC}$, by converse of Thales theorem, $DE \parallel BC$

$\therefore$ $\angle ADE = \angle ABC$ and (1)

$\angle DEA = \angle BCA$ (2)

But, given that $\angle ADE = \angle DEA$ (3)

From (1), (2) and (3), we get $\angle ABC = \angle BCA$

$\therefore$ $AC = AB$ (If opposite angles are equal, then opposite sides are equal).

Thus, $\triangle ABC$ is isosceles.

Example 6.4

The points D, E and F are taken on the sides AB, BC and CA of a $\triangle ABC$ respectively, such that $DE \parallel AC$ and $FE \parallel AB$.

Prove that $\dfrac{AB}{AD} = \dfrac{AC}{FC}$

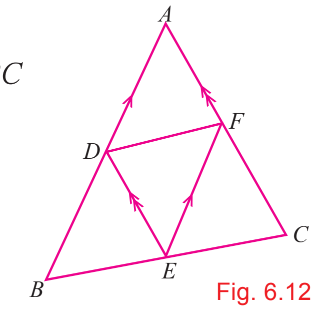


Fig. 6.12

Solution Given that in $\triangle ABC$, $DE \parallel AC$.

$$\therefore \quad \frac{BD}{DA} = \frac{BE}{EC} \quad \text{(Thales theorem)} \qquad (1)$$

Also, given that $FE \parallel AB$.

$$\therefore \quad \frac{BE}{EC} = \frac{AF}{FC} \quad \text{(Thales theorem)} \qquad (2)$$

From (1) and (2), we get

$$\frac{BD}{AD} = \frac{AF}{FC}$$

$$\Longrightarrow \quad \frac{BD+AD}{AD} = \frac{AF+FC}{FC} \quad \text{(componendo rule)}$$

Thus, $\dfrac{AB}{AD} = \dfrac{AC}{FC}$.

Example 6.5

In $\triangle ABC$, the internal bisector AD of $\angle A$ meets the side BC at D. If $BD = 2.5$ cm, $AB = 5$ cm and $AC = 4.2$ cm, then find DC.

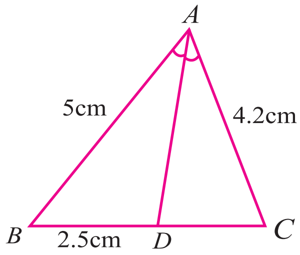


Fig. 6.13

Solution In $\triangle ABC$, AD is the internal bisector of $\angle A$.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} \quad \text{(angle bisector theorem)}$$

$$\Longrightarrow \quad DC = \frac{BD \times AC}{AB}$$

Thus, $DC = \dfrac{2.5 \times 4.2}{5} = 2.1\text{cm}$.

Example 6.6

In $\triangle ABC$, AE is the external bisector of $\angle A$, meeting BC produced at E. If $AB = 10$ cm, $AC = 6$ cm and $BC = 12$ cm, then find CE.

Solution In $\triangle ABC$, AE is the external bisector of $\angle A$ meeting BC produced at E.

Let $CE = x$ cm. Now, by the angle bisector theorem, we have

$$\frac{BE}{CE} = \frac{AB}{AC} \Longrightarrow \frac{12+x}{x} = \frac{10}{6}$$

$$\Longrightarrow 3(12+x) = 5x.$$

$$\Longrightarrow x = 18.$$

Hence, $CE = 18$ cm.

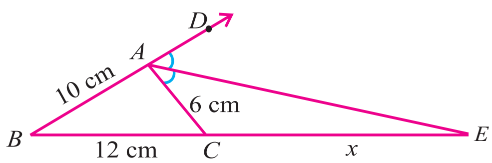


Fig. 6.14

Example 6.7

D is the midpoint of the side BC of $\triangle ABC$. If P and Q are points on AB and on AC such that DP bisects $\angle BDA$ and DQ bisects $\angle ADC$, then prove that $PQ \parallel BC$.

Solution In $\triangle ABD$, DP is the angle bisector of $\angle BDA$.

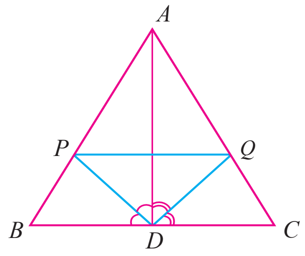


Fig. 6.15

$\therefore \quad \dfrac{AP}{PB} = \dfrac{AD}{BD}$ (angle bisector theorem) (1)

In $\triangle ADC$, DQ is the bisector of $\angle ADC$

$\therefore \quad \dfrac{AQ}{QC} = \dfrac{AD}{DC}$ (angle bisector theorem) (2)

But, $BD = DC$ (D is the midpoint of BC)

Now (2) $\Longrightarrow \dfrac{AQ}{QC} = \dfrac{AD}{BD}$ (3)

From (1) and (3) we get,

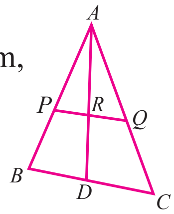
$$\frac{AP}{PB} = \frac{AQ}{QC}$$

Thus, $PQ \parallel BC$. (converse of Thales theorem)

Exercise 6.1

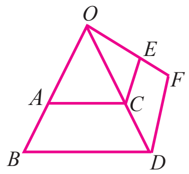
1. In a ΔABC, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.
   (i) If $AD = 6$ cm, $DB = 9$ cm and $AE = 8$ cm, then find AC.
   (ii) If $AD = 8$ cm, $AB = 12$ cm and $AE = 12$ cm, then find CE.
   (iii) If $AD = 4x–3$, $BD = 3x–1$, $AE = 8x–7$ and $EC = 5x–3$, then find the value of x.

2. In the figure, $AP = 3$ cm, $AR = 4.5$ cm, $AQ = 6$ cm, $AB = 5$ cm, and $AC = 10$ cm. Find the length of AD.



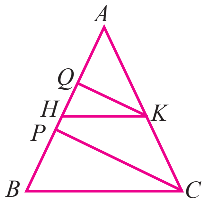
3. E and F are points on the sides PQ and PR respectively, of a $\triangle PQR$. For each of the following cases, verify $EF \parallel QR$.
   (i) $PE = 3.9$ cm, $EQ = 3$ cm, $PF = 3.6$ cm and $FR = 2.4$ cm.
   (ii) $PE = 4$ cm, $QE = 4.5$ cm, $PF = 8$ cm and $RF = 9$ cm.

4. In the figure, $AC \parallel BD$ and $CE \parallel DF$. If $OA = 12$cm, $AB = 9$ cm, $OC = 8$ cm and $EF = 4.5$ cm , then find FO.

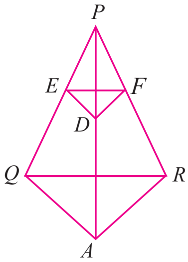


5. $ABCD$ is a quadrilateral with AB parallel to CD. A line drawn parallel to AB meets AD at P and BC at Q. Prove that $\dfrac{AP}{PD} = \dfrac{BQ}{QC}$.

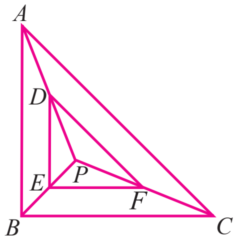
6. In the figure, $PC \parallel QK$ and $BC \parallel HK$. If $AQ = 6$ cm, $QH = 4$ cm, $HP = 5$ cm, $KC = 18$cm, then find AK and PB.



7. In the figure, $DE \parallel AQ$ and $DF \parallel AR$. Prove that $EF \parallel QR$.



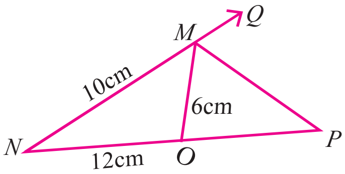
8. In the figure $DE \parallel AB$ and $DF \parallel AC$. Prove that $EF \parallel BC$.



9. In a $\triangle ABC$, AD is the internal bisector of $\angle A$, meeting BC at D.
   (i) If $BD = 2$ cm, $AB = 5$ cm, $DC = 3$ cm find AC.
   (ii) If $AB = 5.6$ cm, $AC = 6$ cm and $DC = 3$ cm find BC.
   (iii) If $AB = x$, $AC = x–2$, $BD = x+2$ and $DC = x–1$ find the value of x.

10. Check whether AD is the bisector of $\angle A$ of $\triangle ABC$ in each of the following.
   (i) $AB = 4$ cm, $AC = 6$ cm, $BD = 1.6$ cm, and $CD = 2.4$ cm.
   (ii) $AB = 6$ cm, $AC = 8$cm, $BD = 1.5$cm and $CD = 3$cm.

11. In a $\triangle MNO$, MP is the external bisector of $\angle M$ meeting NO produced at P. If $MN = 10$ cm, $MO = 6$ cm, $NO = 12$ cm, then find OP.



12. In a quadrilateral $ABCD$, the bisectors of $\angle B$ and $\angle D$ intersect on AC at E. Prove that $\dfrac{AB}{BC} = \dfrac{AD}{DC}$.

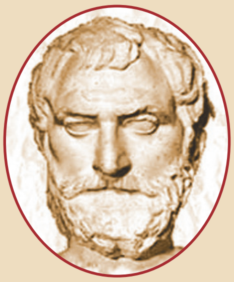
13. The internal bisector of $\angle A$ of ΔABC meets BC at D and the external bisector of $\angle A$ meets BC produced at E. Prove that $\dfrac{BD}{BE} = \dfrac{CD}{CE}$.

14. $ABCD$ is a quadrilateral with $AB = AD$. If AE and AF are internal bisectors of $\angle BAC$ and $\angle DAC$ respectively, then prove that $EF \parallel BD$.

6.3 Similar triangles

In class VIII, we have studied congruence of triangles in detail. We have learnt that two geometrical figures are congruent if they have the same size and shape. In this section, we shall study about those geometrical figures which have the same shape but not necessarily the same size. Such geometrical figures are called similar.

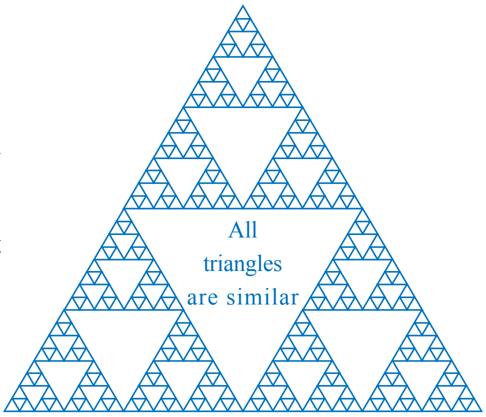
On looking around us, we see many objects which are of the same shape but of same or different sizes. For example, leaves of a tree have almost the same shapes but same or different sizes. Similarly photographs of different sizes developed from the same negative are of same shape but different sizes. All those objects which have the same shape but different sizes are called similar objects.



Thales of Miletus
(624-546 BC)
Greece

Thales was the first known philosopher, scientist and mathematician. He is credited with the first use of deductive reasoning applied to geometry. He discovered many prepositions in geometry. His method of attacking problems invited the attention of many mathematicians. He also predicted an eclipse of the Sun in 585 BC.

Thales said to have introduced Geometry in Greece, is believed to have found the heights of the Pyramids in Egypt, using shadows and the principle of similar triangles. Thus the use of similar triangles has made possible the measurements of heights and distances. He observed that the base angles of an isosceles triangle are equal. He used the idea of similar triangles and right triangles in practical geometry.



It is clear that the congruent figures are similar but the converse need not be true. In this section, we shall discuss only the similarity of triangles and apply this knowledge in solving problems. The following simple activity helps us to visualize similar triangles.

Activity

- Take a cardboard and make a triangular hole in it.
- Expose this cardboard to Sunlight at about one metre above the ground .
- Move it towards the ground to see the formation of a sequence of triangular shapes on the ground.
- Moving close to the ground, the image becomes smaller and smaller. Moving away from the ground, the image becomes larger and larger.
- You see that, the size of the angles forming the three vertices of the triangle would always be the same, even though their sizes are different.

Definition

Two triangles are similar if

(i) their corresponding angles are equal (or)

(ii) their corresponding sides have lengths in the same ratio (or proportional), which is equivalent to saying that one triangle is an enlargement of other.

Thus, two triangles $\triangle ABC$ and $\triangle DEF$ are similar if

(i) $\angle A = \angle D,\ \angle B = \angle E,\ \angle C = \angle F$ (or)

(ii) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$.

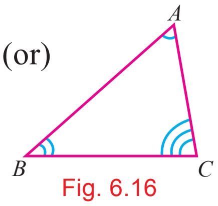


Fig. 6.16

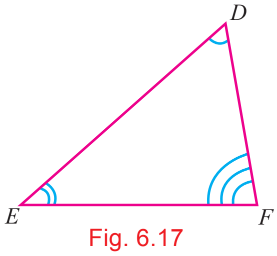


Fig. 6.17

Here, the vertices A, B and C correspond to the vertices D, E and F respectively. Symbolically, we write the similarity of these two triangles as $\triangle ABC \sim \triangle DEF$ and read it as $\triangle ABC$ is similar to $\triangle DEF$. The symbol '$\sim$' stands for 'is similar to'.

Remarks

Similarity of $\triangle ABC$ and $\triangle DEF$ can also be expressed symbolically using correct correspondence of their vertices as $\triangle BCA \sim \triangle EFD$ and $\Delta CAB \sim \Delta FDE$.

6.3.1 Criteria for similarity of triangles

The following three criteria are sufficient to prove that two triangles are similar.

(i) AA(Angle-Angle) similarity criterion

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Remark If two angles of a triangle are respectively equal to two angles of another triangle then their third angles will also be equal. Therefore AA similarity criterion is also referred to AAA criteria.

(ii) SSS (Side-Side-Side) similarity criterion for Two Triangles

In two triangles, if the sides of one triangle are proportional (in the same ratio) to the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

(iii) SAS (Side-Angle-Side) similarity criterion for Two Triangles

If one angle of a triangle is equal to one angle of the other triangle and if the corresponding sides including these angles are proportional, then the two triangles are similar.

Let us list out a few results without proofs on similarity of triangles.

(i) The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

(ii) If a perpendicular is drawn from the vertex of a right angled triangle to its hypotenuse, then the triangles on each side of the perpendicular are similar to the whole triangle.

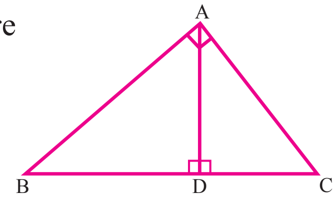


Fig. 6.18

Here, (a) $\Delta\, DBA \sim \Delta\, ABC$

(b) $\Delta\, DAC \sim \Delta\, ABC$

(c) $\Delta\, DBA \sim \Delta\, DAC$

(iii) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of their corresponding altitudes.

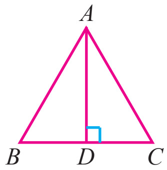


Fig. 6.19

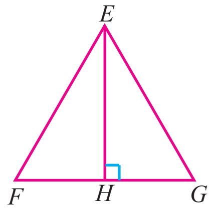


Fig. 6.20

i.e., if $\Delta ABC \sim \Delta EFG$, then $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CA}{GE} = \frac{AD}{EH}$

(iv) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of the corresponding perimeters.

If, $\Delta\, ABC \sim \Delta\, DEF$, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB+BC+CA}{DE+EF+FD}$.

Example 6.8

In ΔPQR, $AB \parallel QR$. If AB is 3 cm, PB is 2cm and PR is 6 cm, then find the length of QR.

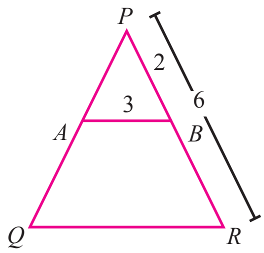


Fig. 6.21

Solution Given AB is 3 cm, PB is 2 cm PR is 6 cm and $AB \parallel QR$

In ΔPAB and ΔPQR

$\angle PAB = \angle PQR$ (corresponding angles)

and $\angle P$ is common.

$\therefore\ \Delta PAB \sim \Delta PQR$ (AA similarity criterion)

Since corresponding sides are proportional,

$$\frac{AB}{QR} = \frac{PB}{PR}$$

$$QR = \frac{AB \times PR}{PB}$$

$$= \frac{3 \times 6}{2}$$

Thus, $QR = 9$ cm.

Example 6.9

A man of height 1.8 m is standing near a Pyramid. If the shadow of the man is of length 2.7 m and the shadow of the Pyramid is 210 m long at that instant, find the height of the Pyramid.

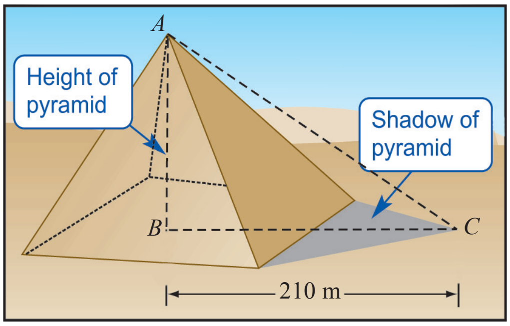


Fig. 6.22

Solution Let AB and DE be the heights of the Pyramid and the man respectively.

Let BC and EF be the lengths of the shadows of the Pyramid and the man respectively.

In ΔABC and ΔDEF, we have

$$\angle ABC = \angle DEF = 90^{\circ}$$
$$\angle BCA = \angle EFD$$

(angular elevation is same at the same instant)

$$\therefore \Delta ABC \sim \Delta DEF \qquad \text{(AA similarity criterion)}$$

Thus, $\dfrac{AB}{DE} = \dfrac{BC}{EF}$

$$\Longrightarrow \frac{AB}{1.8} = \frac{210}{2.7} \Longrightarrow AB = \frac{210}{2.7} \times 1.8 = 140.$$

Hence, the height of the Pyramid is 140 m.

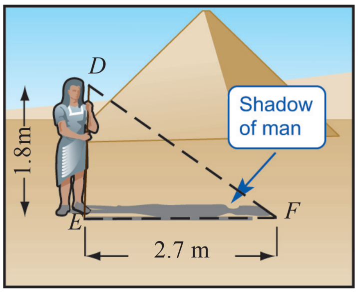


Fig. 6.23

Example 6.10

A man sees the top of a tower in a mirror which is at a distance of 87.6 m from the tower. The mirror is on the ground, facing upward. The man is 0.4 m away from the mirror, and the distance of his eye level from the ground is 1.5 m. How tall is the tower? (The foot of man, the mirror and the foot of the tower lie along a straight line).

Solution Let AB and ED be the heights of the man and the tower respectively. Let C be the point of incidence of the tower in the mirror.

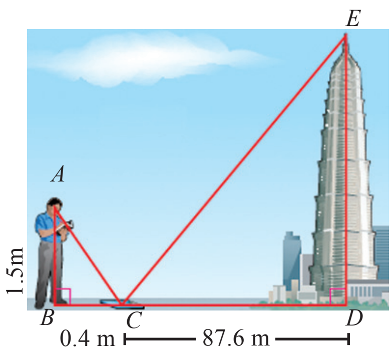


Fig. 6.24

In ΔABC and ΔEDC, we have

$$\angle ABC = \angle EDC = 90^{\circ}$$
$$\angle BCA = \angle DCE$$

(angular elevation is same at the same instant. i.e., the angle of incidence and the angle of reflection are same.)

$$\therefore \qquad \Delta ABC \sim \Delta EDC \qquad \text{(AA similarity criterion)}$$

Thus, $\dfrac{ED}{AB} = \dfrac{DC}{BC}$ (corresponding sides are proportional)

$$ED = \frac{DC}{BC} \times AB = \frac{87.6}{0.4} \times 1.5 = 328.5$$

Hence, the height of the tower is 328.5 m.

Example 6.11

The image of a tree on the film of a camera is of length 35 mm, the distance from the lens to the film is 42 mm and the distance from the lens to the tree is 6 m. How tall is the portion of the tree being photographed?

Solution Let AB and EF be the heights of the portion of the tree and its image on the film respectively.

Let the point C denote the lens.

Let CG and CH be altitudes of ΔACB and ΔFEC.

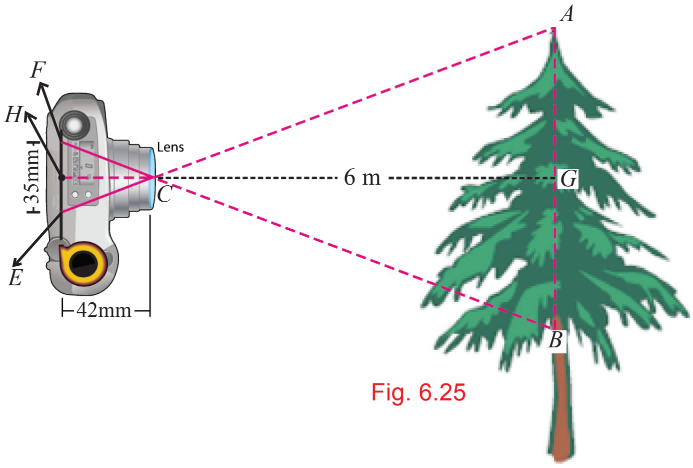


Fig. 6.25

Clearly, AB || FE.

In ΔACB and ΔFEC,

$\angle BAC = \angle FEC$

$\angle ECF = \angle ACB$ (vertically opposite angles)

$\therefore$ $\Delta ACB \sim \Delta ECF$ (AA criterion)

Thus, $\dfrac{AB}{EF} = \dfrac{CG}{CH}$

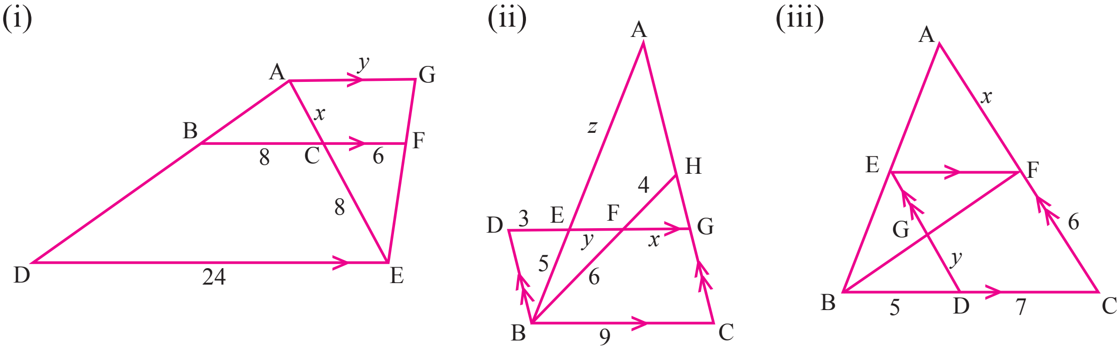
$\Longrightarrow$ $AB = \dfrac{CG}{CH} \times EF = \dfrac{6 \times 0.035}{0.042} = 5.$

Hence, the height of the tree photographed is 5m.

Exercise 6.2

1. Find the unknown values in each of the following figures. All lengths are given in centimetres. (All measures are not in scale)

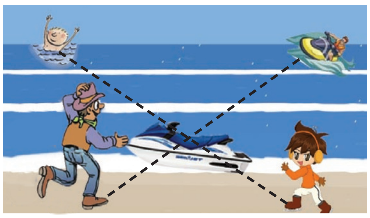
(i) (ii) (iii)



2. The image of a man of height 1.8 m, is of length 1.5 cm on the film of a camera. If the film is 3 cm from the lens of the camera, how far is the man from the camera?

3. A girl of height 120 cm is walking away from the base of a lamp-post at a speed of 0.6 m/sec. If the lamp is 3.6 m above the ground level, then find the length of her shadow after 4 seconds.

4. A girl is in the beach with her father. She spots a swimmer drowning. She shouts to her father who is 50 m due west of her. Her father is 10 m nearer to a boat than the girl. If her father uses the boat to reach the swimmer, he has to travel a distance 126 m from that boat. At the same time, the girl spots a man riding a water craft who is 98 m away from the boat. The man on the water craft is due east of the swimmer. How far must the man travel to rescue the swimmer? (Hint : see figure). (Not for the examination)



5. P and Q are points on sides AB and AC respectively, of ΔABC. If $AP = 3$ cm, $PB = 6$ cm, $AQ = 5$ cm and $QC = 10$ cm, show that $BC = 3\ PQ$.

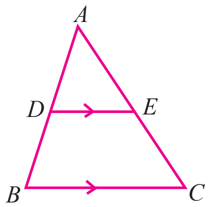
6. In ΔABC, $AB = AC$ and $BC = 6$ cm. D is a point on the side AC such that $AD = 5$ cm and $CD = 4$ cm. Show that $\Delta BCD \sim \Delta ACB$ and hence find BD.

7. The points D and E are on the sides AB and AC *of* ΔABC respectively, such that $DE \parallel BC$. If $AB = 3\ AD$ and the area of ΔABC is 72 cm^2, then find the area of the quadrilateral $DBCE$.

8. The lengths of three sides of a triangle ABC are 6 cm, 4 cm and 9 cm. $\triangle PQR \sim \triangle ABC$. One of the lengths of sides of $\triangle PQR$ is 35cm. What is the greatest perimeter possible for $\triangle PQR$?

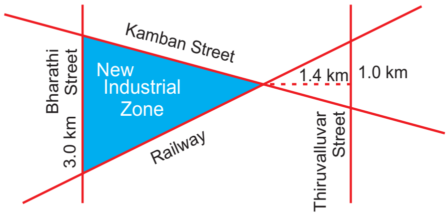
9. In the figure, $DE \parallel BC$ and $\dfrac{AD}{BD} = \dfrac{3}{5}$, calculate the value of

(i) $\dfrac{\text{area of } \Delta ADE}{\text{area of } \Delta ABC}$, (ii) $\dfrac{\text{area of trapezium } BCED}{\text{area of } \Delta ABC}$

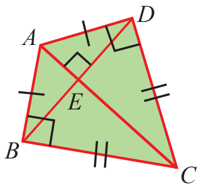


10. The government plans to develop a new industrial zone in an unused portion of land in a city.

The shaded portion of the map shown on the right, indicates the area of the new industrial zone. Find the area of the new industrial zone.

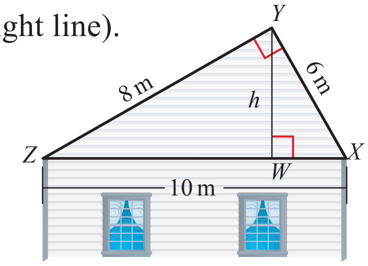


11. A boy is designing a diamond shaped kite, as shown in the figure where $AE = 16$ cm, $EC = 81$ cm. He wants to use a straight cross bar BD. How long should it be?



12. A student wants to determine the height of a flagpole. He placed a small mirror on the ground so that he can see the reflection of the top of the flagpole. The distance of the mirror from him is 0.5 m and the distance of the flagpole from the mirror is 3 m. If his eyes are 1.5 m above the ground level, then find the height of the flagpole.(The foot of student, mirror and the foot of flagpole lie along a straight line).

13. A roof has a cross section as shown in the diagram,
(i) Identify the similar triangles
(ii) Find the height h of the roof.



Theorem 6.5 Pythagoras theorem (Baudhayan theorem)

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Given : In a right angled $\triangle ABC$, $\angle A = 90^\circ$.

To prove : $BC^2 = AB^2 + AC^2$

Construction : Draw $AD \perp BC$

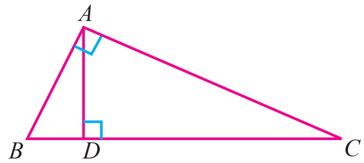


Fig. 6.26

Proof

In triangles ABC and DBA, $\angle B$ is the common angle.

Also, we have $\angle BAC = \angle ADB = 90^\circ$.

$\therefore \triangle ABC \sim \triangle DBA$ (AA similarity criterion)

Thus, their corresponding sides are proportional.

Hence, $\dfrac{AB}{DB} = \dfrac{BC}{BA}$

$\therefore \quad AB^2 = DB \times BC$ (1)

Similarly, we have $\triangle ABC \sim \triangle DAC$.

Thus, $\dfrac{BC}{AC} = \dfrac{AC}{DC}$

$\therefore \quad AC^2 = BC \times DC$ (2)

Adding (1) and (2) we get,

$$AB^2 + AC^2 = BD \times BC + BC \times DC$$
$$= BC(BD + DC)$$
$$= BC \times BC = BC^2$$

Thus, $BC^2 = AB^2 + AC^2$. Hence the Pythagoras theorem.

Remarks

The Pythagoras theorem has two fundamental aspects; one is about areas and the other is about lengths. Hence this landmark thorem connects Geometry and Algebra.The converse of Pythagoras theorem is also true. It was first mentioned and proved by Euclid.

The statement is given below. (Proof is left as an exercise.)

Theorem 6.6 **Converse of Pythagoras theorem**

In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

6.4 Circles and Tangents

A straight line associated with circles is a tangent line which touches the circle at just one point. In geometry, tangent lines to circles play an important role in many geometrical constructions and proofs. . In this section, let us state some results based on circles and tangents and prove an important theorem known as Tangent-Chord thorem. If we consider a straight line and a circle in a plane, then there are three possibilities- they may not intersect at all, they may intersect at two points or they may touch each other at exactly one point. Now look at the following figures.

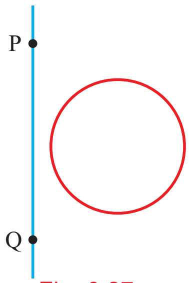


Fig. 6.27

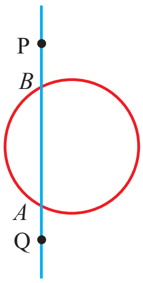


Fig. 6.28

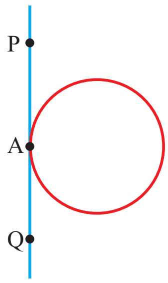


Fig. 6.29

In Fig. 6.27, the circle and the straight line PQ have no common point.

In Fig. 6.28, the straight line PQ cuts the circle at two distinct points A and B. In this case, PQ is called a secant to the circle.

In Fig. 6.29, the straight line PQ and the circle have exactly one common point. Equivalently the straight line touches the circle at only one point. The straight line PQ is called the tangent to the circle at A.

Definition

A straight line which touches a circle at only one point is called a tangent to the circle and the point at which it touches the circle is called its point of contact.

Theorems based on circles and tangents (without proofs)

1. A tangent at any point on a circle is perpendicular to the radius through the point of contact .
2. Only one tangent can be drawn at any point on a circle. However, from an exterior point of a circle two tangents can be drawn to the circle.
3. The lengths of the two tangents drawn from an exterior point to a circle are equal.
4. If two circles touch each other, then the point of contact of the circles lies on the line joining the centres.
5. If two circles touch externally, the distance between their centres is equal to the sum of their radii.
6. If two circles touch internally, the distance between their centres is equal to the difference of their radii.

Theorem 6.7 Tangent-Chord theorem

If from the point of contact of tangent (of a circle), a chord is drawn, then the angles which the chord makes with the tangent line are equal respectively to the angles formed by the chord in the corresponding alternate segments.

Given : O is the centre of the circle. ST is the tangent at A, and AB is a chord. P and Q are any two points on the circle in the opposite sides of the chord AB.

To prove : (i) $\angle BAT = \angle BPA$ (ii) $\angle BAS = \angle AQB$.

Construction: Draw the diameter AC of the circle. Join B and C.

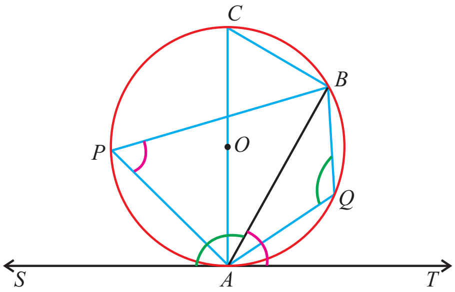


Fig. 6.30

Proof

Statement	Reason	
$\angle ABC = 90^\circ$	angle in a semi-circle is 90°	
$\angle CAB + \angle BCA = 90^\circ$	sum of two acute angles of a right $\triangle ABC$.	(1)
$\angle CAT = 90^\circ$	diameter is $\perp$ to the tangent at the point of contact.	
$\Longrightarrow \angle CAB + \angle BAT = 90^\circ$		(2)
$\angle CAB + \angle BCA = \angle CAB + \angle BAT$	from (1) and (2)	
$\Longrightarrow \quad \angle BCA = \angle BAT$		(3)

$\angle BCA = \angle BPA$ — angles in the same segment (4)

$\angle BAT = \angle BPA$. Hence (i). — from (3) and (4) (5)

Now $\angle BPA + \angle AQB = 180^\circ$ — opposite angles of a cyclic quadrilateral

$\Longrightarrow \angle BAT + \angle AQB = 180^\circ$ — from (5) (6)

Also $\angle BAT + \angle BAS = 180^\circ$ — linear pair (7)

$\angle BAT + \angle AQB = \angle BAT + \angle BAS$ — from (6) and (7)

$\angle BAS = \angle AQB$. Hence (ii).

Thus, the Tangent -Chord theorem is proved.

Theorem 6.8 Converse of Tangent-Chord theorem

If in a circle, through one end of a chord, a straight line is drawn making an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

Definition

Let P be a point on a line segment AB. The product $PA \times PB$ represents the area of the rectangle whose sides are PA and PB.

This product is called the area of the rectangle contained by the parts PA and PB of the line segment AB.

Theorem 6.9

If two chords of a circle intersect either inside or outside of the circle, then the area of the rectangle contained by the segments of one chord is equal to the area of the rectangle contained by the segments of the other chord.

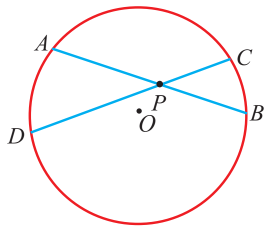


Fig. 6.31

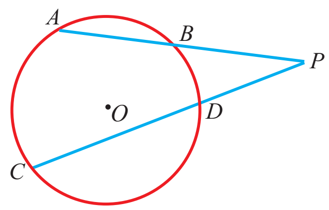


Fig. 6.32

In Fig.6.31, two chords AB and CD intersect at P inside the circle with centre at O. Then $PA \times PB = PC \times PD$. In Fig. 6.32, the chords AB and CD intersect at P outside the circle with centre O. Then $PA \times PB = PC \times PD$.

Example 6.12

Let PQ be a tangent to a circle at A and AB be a chord. Let C be a point on the circle such that $\angle BAC = 54^\circ$ and $\angle BAQ = 62^\circ$. Find $\angle ABC$..

Solution Since PQ is a tangent at A and AB is a chord, we have

$$\angle BAQ = \angle ACB = 62^\circ. \qquad \text{(tangent-chord theorem)}$$

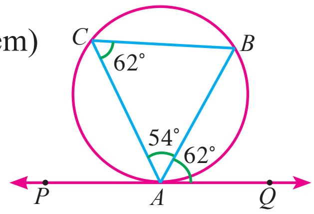


Fig. 6.33

Also, $\angle BAC + \angle ACB + \angle ABC = 180^\circ.$

(sum of all angles in a triangle is 180°)

Thus, $\angle ABC = 180^\circ - (\angle BAC + \angle ACB)$

Hence, $\angle ABC = 180^\circ - (54^\circ + 62^\circ) = 64^\circ.$

Example 6.13

Find the value of x in each of the following diagrams.

(i)

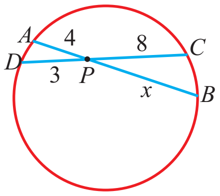


Fig. 6.34

(ii)

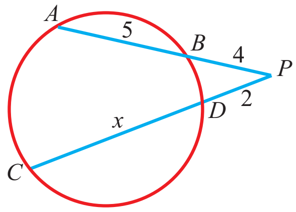


Fig. 6.35

Solution (i) We have $PA \,.\, PB = PC \,.\, PD$

$$PB = \frac{PC.PD}{PA}$$

Thus, $x = \dfrac{8 \times 3}{4} = 6.$

(ii) We have $PC \,.\, PD = PA.\, PB$

$$(2+x)\, 2 = 9 \times 4$$

$x + 2 = 18$. Thus, $x = 16$.

Example 6.14

In the figure, tangents PA and PB are drawn to a circle with centre O from an external point P. If CD is a tangent to the circle at E and $AP = 15$ cm, find the perimeter of ΔPCD

Solution We know that the lengths of the two tangents from an exterior point to a circle are equal.

$\therefore \; CA = CE, \; DB = DE$ and $PA = PB$.

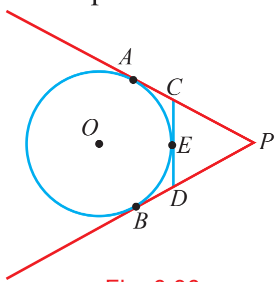


Fig. 6.36

Now, the perimeter of $\Delta PCD = PC + CD + DP$

$$= PC + CE + ED + DP$$

$$= PC + CA + DB + DP$$

$$= PA + PB = 2\,PA \qquad (PB = PA)$$

Thus, the perimeter of $\Delta PCD = 2 \times 15 = 30$ cm.

Example 6.15

$ABCD$ is a quadrilateral such that all of its sides touch a circle. If $AB = 6$ cm, $BC = 6.5$ cm and $CD = 7$ cm , then find the length of AD.

Solution Let P, Q, R and S be the points where the circle touches the quadrilateral. We know that the lengths of the two tangents drawn from an exterior point to a circle are equal.

Thus, we have, $AP = AS$, $BP = BQ$, $CR = CQ$ and $DR = DS$.

Hence, $AP + BP + CR + DR = AS + BQ + CQ + DS$

$\Longrightarrow \quad AB + CD = AD + BC.$

$\Longrightarrow \quad AD = AB + CD - BC$

$= 6 + 7 - 6.5 = 6.5$

Thus, $AD = 6.5$ cm.

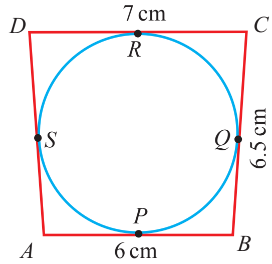
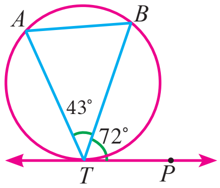


Fig. 6.37

Exercise 6.3



1. In the figure TP is a tangent to a circle. A and B are two points on the circle. If $\angle BTP = 72^\circ$ and $\angle ATB = 43^\circ$ find $\angle ABT$.
2. AB and CD are two chords of a circle which intersect each other internally at P. (i) If $CP = 4$ cm, $AP = 8$ cm, $PB = 2$ cm, then find PD.

   (ii) If $AP = 12$ cm, $AB = 15$ cm, $CP = PD$, then find CD
3. AB and CD are two chords of a circle which intersect each other externally at P

   (i) If $AB = 4$ cm $BP = 5$ cm and $PD = 3$ cm, then find CD.

   (ii) If $BP = 3$ cm, $CP = 6$ cm and $CD = 2$ cm, then find AB
4. A circle touches the side BC of ΔABC at P, AB and AC produced at Q and R respectively, prove that $AQ = AR = \frac{1}{2}$(perimeter of ΔABC)
5. If all sides of a parallelogram touch a circle, show that the parallelogram is a rhombus.
6. A lotus is 20 cm above the water surface in a pond and its stem is partly below the water surface. As the wind blew, the stem is pushed aside so that the lotus touched the water 40 cm away from the original position of the stem. How much of the stem was below the water surface originally?
7. A point O in the interior of a rectangle $ABCD$ is joined to each of the vertices A, B, C and D. Prove that $OA^2 + OC^2 = OB^2 + OD^2$

Exercise 6.4

Choose the correct answer

1. If a straight line intersects the sides AB and AC of a $\triangle ABC$ at D and E respectively and is parallel to BC, then $\frac{AE}{AC} =$

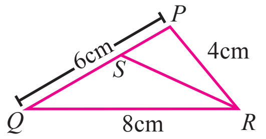
   (A) $\frac{AD}{DB}$ (B) $\frac{AD}{AB}$ (C) $\frac{DE}{BC}$ (D) $\frac{AD}{EC}$

2. In $\triangle ABC$, DE is $\parallel$ to BC, meeting AB and AC at D and E.

If $AD = 3$ cm, $DB = 2$ cm and $AE = 2.7$ cm, then AC is equal to

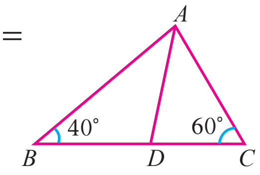
(A) 6.5 cm (B) 4.5 cm (C) 3.5 cm (D) 5.5 cm

3. In $\triangle PQR$, RS is the bisector of $\angle R$. If $PQ = 6$ cm, $QR = 8$ cm, $RP = 4$ cm then PS is equal to



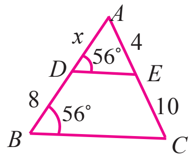
(A) 2 cm (B) 4 cm (C) 3 cm (D) 6 cm

4. In figure, if $\dfrac{AB}{AC} = \dfrac{BD}{DC}$, $\angle B = 40^\circ$, and $\angle C = 60^\circ$, then $\angle BAD =$



(A) 30° (B) 50° (C) 80° (D) 40°

5. In the figure, the value x is equal to



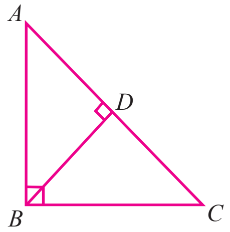
(A) $4 \cdot 2$ (B) $3 \cdot 2$

(C) $0 \cdot 8$ (D) $0 \cdot 4$

6. In triangles ABC and DEF, $\angle B = \angle E$, $\angle C = \angle F$, then

(A) $\dfrac{AB}{DE} = \dfrac{CA}{EF}$ (B) $\dfrac{BC}{EF} = \dfrac{AB}{FD}$ (C) $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ (D) $\dfrac{CA}{FD} = \dfrac{AB}{EF}$

7. From the given figure, identify the wrong statement.



(A) $\Delta ADB \sim \Delta ABC$ (B) $\Delta ABD \sim \Delta ABC$

(C) $\Delta BDC \sim \Delta ABC$ (D) $\Delta ADB \sim \Delta BDC$

8. If a vertical stick 12 m long casts a shadow 8 m long on the ground and at the same time a tower casts a shadow 40 m long on the ground, then the height of the tower is

(A) 40 m (B) 50 m (C) 75 m (D) 60 m

9. The sides of two similar triangles are in the ratio 2:3, then their areas are in the ratio

(A) 9:4 (B) 4:9 (C) 2:3 (D) 3:2

10. Triangles ABC and DEF are similar. If their areas are 100 cm^2 and 49 cm^2 respectively and BC is 8.2 cm then EF =

(A) 5.47 cm (B) 5.74 cm (C) 6.47 cm (D) 6.74 cm

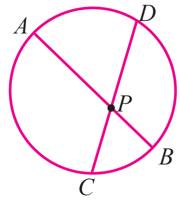
11. The perimeters of two similar triangles are 24 cm and 18 cm respectively. If one side of the first triangle is 8 cm, then the corresponding side of the other triangle is

(A) 4 cm (B) 3 cm (C) 9 cm (D) 6 cm

12. AB and CD are two chords of a circle which when produced to meet at a point P such that $AB = 5$ cm, $AP = 8$ cm, and $CD = 2$ cm then $PD =$

(A) 12 cm (B) 5 cm (C) 6 cm (D) 4 cm

13. In the adjoining figure, chords AB and CD intersect at P. If $AB = 16$ cm, $PD = 8$ cm, $PC = 6$ and AP >PB, then AP =

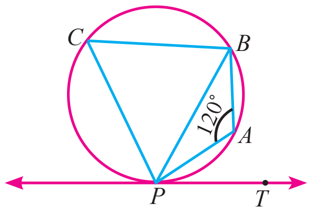


(A) 8 cm (B) 4 cm (C) 12 cm (D) 6 cm

14. A point P is 26 cm away from the centre O of a circle and PT is the tangent drawn from P to the circle is 10 cm, then OT is equal to

(A) 36 cm (B) 20 cm (C) 18 cm (D) 24 cm

15. In the figure, if $\angle PAB = 120^\circ$ then $\angle BPT =$

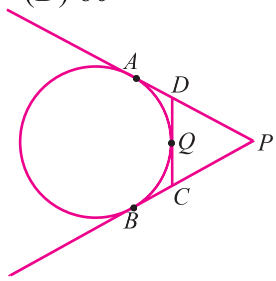


(A) 120° (B) 30° (C) 40° (D) 60°

16. If the tangents PA and PB from an external point P to circle with centre O are inclined to each other at an angle of 40°, then $\angle POA =$

(A) 70° (B) 80° (C) 50° (D) 60°

17. In the figure, PA and PB are tangents to the circle drawn from an external point P. Also CD is a tangent to the circle at Q. If $PA = 8$ cm and $CQ = 3$ cm, then PC is equal to



(A) 11 cm (B) 5 cm (C) 24 cm (D) 38 cm

18. ΔABC is a right angled triangle where $\angle B = 90^\circ$ and $BD \perp AC$. If BD = 8 cm, $AD = 4$ cm, then CD is

(A) 24 cm (B) 16 cm (C) 32 cm (D) 8 cm

19. The areas of two similar triangles are 16 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 3 cm, then the corresponding altitude of the other triangle is

(A) 6.5 cm (B) 6 cm (C) 4 cm (D) 4.5 cm

20. The perimeter of two similar triangles ΔABC and ΔDEF are 36 cm and 24 cm respectively. If $DE = 10$ cm, then AB is

(A) 12 cm (B) 20 cm (C) 15 cm (D) 18 cm

7

- Introduction
- Identities
- Heights and Distances

Hipparchus
(190 - 120 B.C.)
Greece

Hipparchus developed trigonometry, constructed trigonometric tables and solved several problems of spherical trigonometry. With his solar and lunar theories and his trigonometry, he may have been the first to develop a reliable method to predict solar eclipses.

Hipparchus is credited with the invention or improvement of several astronomical instruments, which were used for a long time for naked-eye observations.

TRIGONOMETRY

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry – J.F. Herbart

7.1 Introduction

Trigonometry was developed to express relationship between the sizes of arcs in circles and the chords determining those arcs. After 15^{th} century it was used to relate the measure of angles in a triangle to the lengths of the sides of the triangle. The creator of Trigonometry is said to have been the Greek Hipparchus of the second century B.C. The word **Trigonometry** which means triangle measurement, is credited to Bartholomaus Pitiscus (1561-1613).

We have learnt in class IX about various trigonometric ratios, relation between them and how to use trigonometric tables in solving problems.

In this chapter, we shall learn about trigonometric identities, application of trigonometric ratios in finding heights and distances of hills, buildings etc., without actually measuring them.

7.2 Trigonometric identities

We know that an equation is called an identity when it is true for all values of the variable(s) for which the equation is meaningful. For example, the equation $(a+b)^2 = a^2 + 2ab + b^2$ is an identity since it is true for all real values of a and b.

Likewise, an equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved in the equation. For example, the equation $(\sin\theta + \cos\theta)^2 - (\sin\theta - \cos\theta)^2 = 4\sin\theta\cos\theta$ is a trigonometric identity as it is true for all values of θ.

However, the equation $(\sin\theta + \cos\theta)^2 = 1$ is not an identity because it is true when $\theta = 0^\circ$, but not true when $\theta = 45^\circ$ as $(\sin 45^\circ + \cos 45^\circ)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 = 2 \neq 1$.

In this chapter, all the trigonometric identities and equations are assumed to be well defined for those values of the variables for which they are meaningful.

Let us establish three useful identities called the Pythagorean identities and use them to obtain some other identities.

In the right-angled $\triangle ABC$, we have

$$AB^2 + BC^2 = AC^2 \qquad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2} \qquad (AC \neq 0)$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

Thus, $\cos^2 A + \sin^2 A = 1$

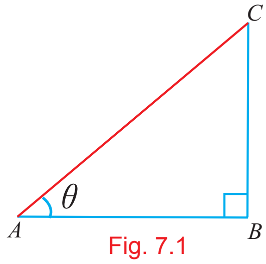


Fig. 7.1

Let $\angle A = \theta$. Then for all $0^\circ < \theta < 90^\circ$ we have,

$$\cos^2\theta + \sin^2\theta = 1. \qquad (2)$$

Evidently, $\cos^2 0^\circ + \sin^2 0^\circ = 1$ and $\cos^2 90^\circ + \sin^2 90^\circ = 1$ and so (2) is true for all θ such that $0^\circ \leq \theta \leq 90^\circ$

Let us divide (1) by AB^2, we get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \left(\frac{AC}{AB}\right)^2 \qquad (\because AB \neq 0)$$

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2 \implies 1 + \tan^2\theta = \sec^2\theta. \qquad (3)$$

Since $\tan\theta$ and $\sec\theta$ are not defined for $\theta = 90^\circ$, the identity (3) is true for all θ such that $0^\circ \leq \theta < 90^\circ$

Again dividing each term of (1) by BC^2, we get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \left(\frac{AC}{BC}\right)^2 \qquad (\because BC \neq 0)$$

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2 \implies \cot^2\theta + 1 = \operatorname{cosec}^2\theta. \qquad (4)$$

Since $\cot\theta$ and $\operatorname{cosec}\theta$ are not defined for $\theta = 0^\circ$, the identity (4) is true for all θ such that $0^\circ < \theta \leq 90^\circ$

Some equal forms of identities from (2) to (4) are listed below.

	Identity	Equal forms
(i)	$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
(ii)	$1 + \tan^2\theta = \sec^2\theta$	$\sec^2\theta - \tan^2\theta = 1$ (or) $\tan^2\theta = \sec^2\theta - 1$
(iii)	$1 + \cot^2\theta = \text{cosec}^2\theta$	$\text{cosec}^2\theta - \cot^2\theta = 1$ (or) $\cot^2\theta = \text{cosec}^2\theta - 1$

Remarks

We have proved the above identities for an acute angle θ. But these identities are true for any angle θ for which the trigonometric functions are meaningful. In this book we shall restrict ourselves to acute angles only.

In general, there is no common method for proving trigonometric identities involving trigonometric functions. However, some of the techniques listed below may be useful in proving trigonometric identities.

(i) Study the identity carefully, keeping in mind what is given and what you need to arrive.

(ii) Generally, the more complicated side of the identity may be taken first and simplified as it is easier to simplify than to expand or enlarge the simpler one.

(iii) If both sides of the identity are complicated, each may be taken individually and simplified independently of each other to the same expression.

(iv) Combine fractions using algebraic techniques for adding expressions.

(v) If necessary, change each term into their sine and cosine equivalents and then try to simplify.

(vi) If an identity contains terms involving $\tan^2\theta, \cot^2\theta, \text{cosec}^2\theta, \sec^2\theta$, it may be more helpful to use the results $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$.

Example 7.1

Prove the identity $\dfrac{\sin\theta}{\text{cosec}\,\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

Solution

Now, $\dfrac{\sin\theta}{\text{cosec}\,\theta} + \dfrac{\cos\theta}{\sec\theta} = \dfrac{\sin\theta}{\left(\dfrac{1}{\sin\theta}\right)} + \dfrac{\cos\theta}{\left(\dfrac{1}{\cos\theta}\right)}$

$$= \sin^2\theta + \cos^2\theta = 1\,.$$

Example 7.2

Prove the identity $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$

Solution

Consider $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\dfrac{(1-\cos\theta)}{(1+\cos\theta)}\times\dfrac{(1-\cos\theta)}{(1-\cos\theta)}}$

$$= \sqrt{\frac{(1-\cos\theta)^2}{1^2-\cos^2\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \qquad (1-\cos^2\theta = \sin^2\theta)$$

$$= \frac{1-\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \operatorname{cosec}\theta - \cot\theta\,.$$

Example 7.3

Prove the identity $[\operatorname{cosec}(90^\circ-\theta)-\sin(90^\circ-\theta)][\operatorname{cosec}\theta-\sin\theta][\tan\theta+\cot\theta] = 1$

Solution

Now, $[\operatorname{cosec}(90^\circ-\theta)-\sin(90^\circ-\theta)][\operatorname{cosec}\theta-\sin\theta][\tan\theta+\cot\theta]$

$$= (\sec\theta-\cos\theta)(\operatorname{cosec}\theta-\sin\theta)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) \qquad \begin{matrix}\because\ \operatorname{cosec}(90^\circ-\theta) = \sec\theta\\ \because\ \sin(90^\circ-\theta) = \cos\theta\end{matrix}$$

$$= \left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$= \left(\frac{\sin^2\theta}{\cos\theta}\right)\left(\frac{\cos^2\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta\cos\theta}\right) = 1$$

Example 7.4

Prove that $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{1+\sin\theta}{\cos\theta}$

Solution

We consider $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$$= \frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1} \qquad (\sec^2\theta-\tan^2\theta = 1)$$

$$= \frac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\tan\theta-\sec\theta+1} \qquad (a^2-b^2 = (a+b)(a-b))$$

$$= \frac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$$

$$= \frac{(\tan\theta+\sec\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1}$$

$$= \tan\theta+\sec\theta = \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

Example 7.5

Prove the identity $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\tan\theta+\cot\theta$.

Solution

Now, $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}$

$$=\frac{\tan\theta}{1-\frac{1}{\tan\theta}}+\frac{\frac{1}{\tan\theta}}{1-\tan\theta}=\frac{\tan\theta}{\frac{\tan\theta-1}{\tan\theta}}+\frac{\frac{1}{\tan\theta}}{1-\tan\theta}$$

$$=\frac{\tan^2\theta}{\tan\theta-1}+\frac{1}{\tan\theta(1-\tan\theta)}=\frac{\tan^2\theta}{\tan\theta-1}+\frac{1}{(-\tan\theta)(\tan\theta-1)}$$

$$=\frac{\tan^2\theta}{\tan\theta-1}-\frac{1}{(\tan\theta)(\tan\theta-1)}$$

$$=\frac{1}{(\tan\theta-1)}\left(\tan^2\theta-\frac{1}{\tan\theta}\right)$$

$$=\frac{1}{(\tan\theta-1)}\;\frac{(\tan^3\theta-1)}{\tan\theta}$$

$$=\frac{(\tan\theta-1)(\tan^2\theta+\tan\theta+1^2)}{(\tan\theta-1)\tan\theta}\qquad(\because a^3-b^3=(a-b)(a^2+ab+b^2))$$

$$=\frac{\tan^2\theta+\tan\theta+1}{\tan\theta}$$

$$=\frac{\tan^2\theta}{\tan\theta}+\frac{\tan\theta}{\tan\theta}+\frac{1}{\tan\theta}=\tan\theta+1+\cot\theta$$

$$=1+\tan\theta+\cot\theta.$$

Example 7.6

Prove the identity

$$(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2=7+\tan^2\theta+\cot^2\theta.$$

Solution

Let us consider $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2$

$$=\sin^2\theta+\operatorname{cosec}^2\theta+2\sin\theta\operatorname{cosec}\theta+\cos^2\theta+\sec^2\theta+2\cos\theta\sec\theta$$

$$=\sin^2\theta+\cos^2\theta+\operatorname{cosec}^2\theta+\sec^2\theta+2\sin\theta\frac{1}{\sin\theta}+2\cos\theta\frac{1}{\cos\theta}$$

$$=1+(1+\cot^2\theta)+(1+\tan^2\theta)+2+2$$

$$=7+\tan^2\theta+\cot^2\theta.$$

Example 7.7

Prove the identity $(\sin^6\theta + \cos^6\theta) = 1 - 3\sin^2\theta\cos^2\theta$.

Solution

Now $\sin^6\theta + \cos^6\theta$

$$= (\sin^2\theta)^3 + (\cos^2\theta)^3$$

$$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta\,(\sin^2\theta + \cos^2\theta)$$

$$(\,a^3 + b^3 = (a+b)^3 - 3ab(a+b)\,)$$

$$= 1 - 3\sin^2\theta\cos^2\theta. \qquad (\,\sin^2\theta + \cos^2\theta = 1\,)$$

Example 7.8

Prove the identity $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$.

Solution

Now, $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \dfrac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$

$$= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta + \cos^2\theta - 2\sin^2\theta}{2\cos^2\theta - (\sin^2\theta + \cos^2\theta)}\right) \qquad (\sin^2\theta + \cos^2\theta = 1)$$

$$= (\,\tan\theta\,)\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta - \sin^2\theta}\right) = \tan\theta.$$

Example 7.9

Prove the identity $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$.

Solution

We consider $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$

$$= \left(\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}\right) \times \left(\frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}\right)$$

$$= \frac{(\sec\theta - \tan\theta)^2}{\sec^2\theta - \tan^2\theta}$$

$$= \frac{(\sec\theta - \tan\theta)^2}{1} \qquad (\sec^2\theta - \tan^2\theta = 1)$$

$$= (\sec\theta - \tan\theta)^2 = \sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta$$

$$= (1 + \tan^2\theta) + \tan^2\theta - 2\sec\theta\tan\theta \qquad (\sec^2\theta = 1 + \tan^2\theta)$$

$$= 1 - 2\sec\theta\tan\theta + 2\tan^2\theta.$$

Example 7.10

Prove that $\dfrac{1+\sec\theta}{\sec\theta} = \dfrac{\sin^2\theta}{1-\cos\theta}$.

Solution

First, we consider $\dfrac{1+\sec\theta}{\sec\theta}$

$$= \frac{1+\dfrac{1}{\cos\theta}}{\dfrac{1}{\cos\theta}} = \frac{(\cos\theta+1)}{\cos\theta}(\cos\theta)$$

$$= 1+\cos\theta$$

$$= (1+\cos\theta)\times\frac{(1-\cos\theta)}{(1-\cos\theta)}$$

$$= \frac{1-\cos^2\theta}{1-\cos\theta}$$

$$= \frac{\sin^2\theta}{1-\cos\theta}.$$

Example 7.11

Prove the identity $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta) = \dfrac{1}{\tan\theta+\cot\theta}$.

Solution

Now, $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)$

$$= \left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)$$

$$= \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin\theta}\,\frac{\sin^2\theta}{\cos\theta} = \sin\theta\cos\theta \qquad (1)$$

Next, consider $\dfrac{1}{\tan\theta+\cot\theta}$

$$= \frac{1}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\left(\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)}$$

$$= \sin\theta\cos\theta \qquad (2)$$

Note

$\sin\theta\cos\theta$

$$= \frac{\sin\theta\cos\theta}{1}$$

$$= \frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}$$

$$= \frac{1}{\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{1}{\dfrac{\sin^2\theta}{\sin\theta\cos\theta}+\dfrac{\cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{1}{\tan\theta+\cot\theta}$$

From (1) and (2), we get

$$(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta) = \frac{1}{\tan\theta+\cot\theta}.$$

Example 7.12

If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$ and $m \neq n$, then show that $m^2 - n^2 = 4\sqrt{mn}$.

Solution

Given that $\quad m = \tan\theta + \sin\theta$ and $n = \tan\theta - \sin\theta$.

$$\text{Now,}\quad m^2 - n^2 = (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2$$

$$= \tan^2\theta + \sin^2\theta + 2\sin\theta\tan\theta - (\tan^2\theta + \sin^2\theta - 2\sin\theta\tan\theta)$$

$$= 4\sin\theta\tan\theta \qquad (1)$$

$$\text{Also,}\quad 4\sqrt{mn} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)}$$

$$= 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sqrt{\left(\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta\right)}$$

$$= 4\sqrt{\sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right)}$$

$$= 4\sqrt{\sin^2\theta(\sec^2\theta - 1)} = 4\sqrt{\sin^2\theta\tan^2\theta} \quad (\because \sec^2\theta - 1 = \tan^2\theta)$$

$$= 4\sin\theta\tan\theta. \qquad (2)$$

From (1) and (2), we get $m^2 - n^2 = 4\sqrt{mn}$.

Example 7.13

If $\tan^2\alpha = \cos^2\beta - \sin^2\beta$, then prove that $\cos^2\alpha - \sin^2\alpha = \tan^2\beta$.

Solution

Given that $\quad \cos^2\beta - \sin^2\beta = \tan^2\alpha$

$$\frac{\cos^2\beta - \sin^2\beta}{1} = \frac{\sin^2\alpha}{\cos^2\alpha}$$

$$\frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + \sin^2\beta} = \frac{\sin^2\alpha}{\cos^2\alpha}$$

> Componendo and dividendo rule
>
> If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Applying componendo and dividendo rule, we get

$$\frac{(\cos^2\beta - \sin^2\beta) + (\cos^2\beta + \sin^2\beta)}{(\cos^2\beta - \sin^2\beta) - (\cos^2\beta + \sin^2\beta)} = \frac{\sin^2\alpha + \cos^2\alpha}{\sin^2\alpha - \cos^2\alpha}$$

$$\Longrightarrow \quad \frac{2\cos^2\beta}{-2\sin^2\beta} = \frac{1}{\sin^2\alpha - \cos^2\alpha}$$

$$\Longrightarrow \quad -\frac{\sin^2\beta}{\cos^2\beta} = \sin^2\alpha - \cos^2\alpha$$

$\Longrightarrow \quad \tan^2\beta = \cos^2\alpha - \sin^2\alpha$, which completes the proof.

Note: This problem can also be solved without using componendo and dividendo rule.

Exercise 7.1

1. Determine whether each of the following is an identity or not.

   (i) $\cos^2\theta + \sec^2\theta = 2+\sin\theta$ (ii) $\cot^2\theta + \cos\theta = \sin^2\theta$

2. Prove the following identities

   (i) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$ (ii) $\dfrac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$

   (iii) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (iv) $\dfrac{\cos\theta}{\sec\theta - \tan\theta} = 1 + \sin\theta$

   (v) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$ (vi) $\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$

   (vii) $\sec\theta(1-\sin\theta)(\sec\theta + \tan\theta) = 1$ (viii) $\dfrac{\sin\theta}{\operatorname{cosec}\theta + \cot\theta} = 1 - \cos\theta$

3. Prove the following identities.

   (i) $\dfrac{\sin(90^\circ - \theta)}{1+\sin\theta} + \dfrac{\cos\theta}{1-\cos(90^\circ - \theta)} = 2\sec\theta$

   (ii) $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\operatorname{cosec}\theta$

   (iii) $\dfrac{\sin(90^0 - \theta)}{1-\tan\theta} + \dfrac{\cos(90^0 - \theta)}{1-\cot\theta} = \cos\theta + \sin\theta$

   (iv) $\dfrac{\tan(90^0 - \theta)}{\operatorname{cosec}\theta + 1} + \dfrac{\operatorname{cosec}\theta + 1}{\cot\theta} = 2\sec\theta.$

   (v) $\dfrac{\cot\theta + \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1} = \operatorname{cosec}\theta + \cot\theta.$

   (vi) $(1+\cot\theta - \operatorname{cosec}\theta)(1+\tan\theta + \sec\theta) = 2$

   (vii) $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$

   (viii) $\dfrac{\tan\theta}{1-\tan^2\theta} = \dfrac{\sin\theta\sin(90^0 - \theta)}{2\sin^2(90^0 - \theta) - 1}$

   (ix) $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}.$

   (x) $\dfrac{\cot^2\theta + \sec^2\theta}{\tan^2\theta + \operatorname{cosec}^2\theta} = (\sin\theta\cos\theta)(\tan\theta + \cot\theta).$

4. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, then prove that $x^2 - y^2 = a^2 - b^2$.

5. If $\tan\theta = n\tan\alpha$ and $\sin\theta = m\sin\alpha$, then prove that $\cos^2\theta = \dfrac{m^2-1}{n^2-1}$, $n \neq \pm 1$.

6. If $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P., then prove that $\cot^6\theta - \cot^2\theta = 1$.

7.3 Heights and Distances

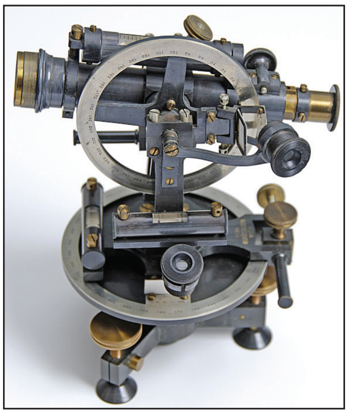


Fig. 7.2

One wonders, how the distance between planets, height of Mount Everest, distance between two objects which are far off like Earth and Sun …, are measured or calculated. Can these be done with measuring tapes?

Of course, it is impossible to do so. Quite interestingly such distances are calculated using the idea of trigonometric ratios. These ratios are also used to construct maps, determine the position of an Island in relation to longitude and latitude.

A theodolite (Fig. 7.2) is an instrument which is used in measuring the angle between an object and the eye of the observer. A theodolite consists of two graduated wheels placed at right angles to each other and a telescope. The wheels are used for the measurement of horizontal and vertical angles. The angle to the desired point is measured by positioning the telescope towards that point. The angle can be read on the telescopic scale.

Suppose we wish to find the height of our school flag post without actually measuring it.

Assume that a student stands on the ground at point *A,* which is 10 m away from the foot *B* of the flag post. He observes the top of the flag post at an angle of 60°. Suppose that the height of his eye level *E* from the ground level is $1.2\,\text{m}$. (see fig no.7.3)

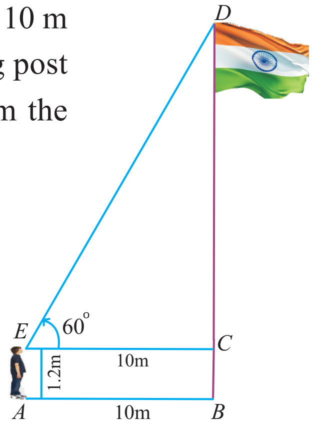


Fig. 7.3

In the right angled ΔDEC, $\angle DEC = 60^\circ$.

Now, $\tan 60^\circ = \dfrac{CD}{EC}$

$\Longrightarrow$ $CD = EC \tan 60^\circ$

Thus, $CD = 10\sqrt{3} = 10 \times 1.732$

$= 17.32\,\text{m}$

Hence, the height of the flag post, $BD = BC + CD$

$= 1.2 + 17.32 = 18.52\,\text{m}$

Thus, we are able to find the height of our school flag post without actually measuring it. So, in a right triangle, if one side and one acute angle are known, we can find the other sides of the triangle using trigonometrical ratios. Let us define a few terms which we use very often in finding the heights and distances.

Line of sight

If we are viewing an object, the line of sight is a straight line from our eye to the object. Here we treat the object as a point since distance involved is quite large.

Angle of depression and angle of elevation

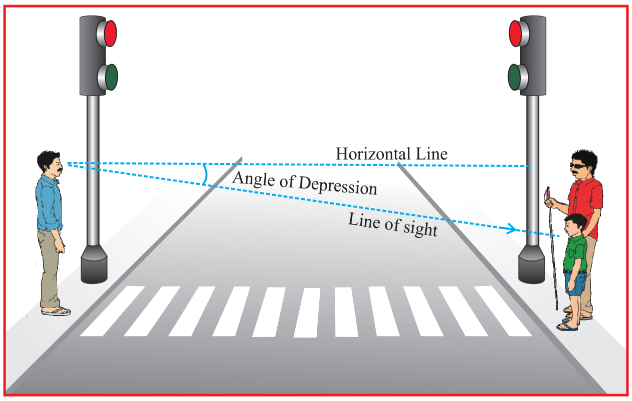


Fig. 7.4

If an object is below the horizontal line from the eye, we have to lower our head to view the object. In this process our eyes moves through an angle. This angle is called the angle of depression, That is, the angle of depression of an object viewed is the angle formed by the line of sight with the horizontal line, when the object is below the horizontal line (See Fig. 7.4).

If an object is above the horizontal line from our eyes we have to raise our head to view the object. In this process our eyes move through an angle formed by the line of sight and horizontal line which is called the angle of elevation. (See Fig. 7.5).

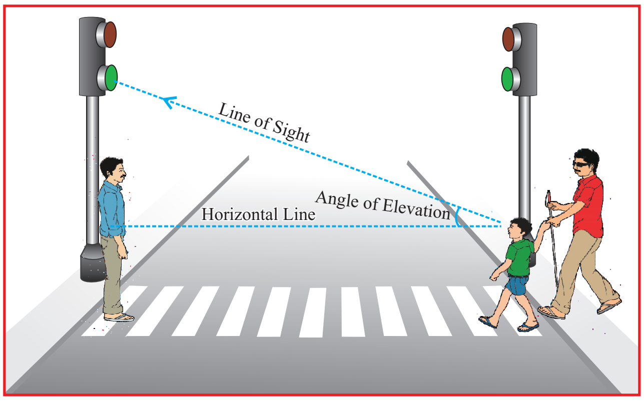


Fig. 7.5

Note

(i) An observer is taken as a point if the height of the observer is not given.

(ii) The angle of elevation of an object as seen by the observer is same as the angle of depression of the observer as seen from the object.

To solve problems involving heights and distances, the following strategy may be useful

(i) Read the statements of the question carefully and draw a rough diagram accordingly.

(ii) Label the diagram and mark the given values.

(iii) Denote the unknown dimension, say 'h' when the height is to be calculated and 'x' when the distance is to be calculated.

(iv) Identify the trigonometrical ratio that will be useful for solving the problem.

(v) Substitute the given values and solve for unknown.

The following activity may help us learn how to measure the height of an object which will be difficult to measure otherwise.

Activity

- Tie one end of a string to the middle of a straw and the other end of the string to a paper clip.
- Glue this straw to the base of a protractor so that the middle of the straw aligns with the centre of the protractor. Make sure that the string hangs freely to create a vertical line or the plumb-line.
- Find an object outside that is too tall to measure directly, such as a basket ball hoop, a flagpole, or the school building.
- Look at the top of the object through the straw. Find the angle where the string and protractor intersect. Determine the angle of elevation by subtracting this measurement from 90°. Let it be θ.
- Measure the distance from your eye level to the ground and from your foot to the base of the object that you are measuring, say y.
- Make a sketch of your measurements.
- To find the height (h) of the object, use the following equation.

  $h = x + y\tan\theta$, where x represents the distance from your eye level to the ground.

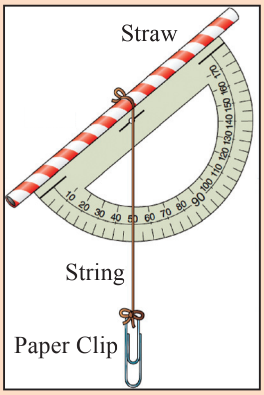


Fig. 7.6

Example 7.14

A kite is flying with a string of length 200 m. If the thread makes an angle 30° with the ground, find the distance of the kite from the ground level. (Here, assume that the string is along a straight line)

Solution Let h denote the distance of the kite from the ground level.

In the figure, AC is the string

Given that $\angle CAB = 30^\circ$ and $AC = 200$ m.

In the right $\triangle$ CAB, $\sin 30^\circ = \dfrac{h}{200}$

$$\Longrightarrow \quad h = 200\sin 30^\circ$$

$$\therefore \quad h = 200 \times \frac{1}{2} = 100 \text{ m}$$

Hence, the distance of the kite from the ground level is 100 m.

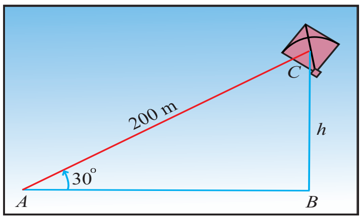


Fig. 7.7

Example 7.15

A ladder leaning against a vertical wall, makes an angle of 60° with the ground. The foot of the ladder is 3.5 m away from the wall. Find the length of the ladder.

Solution Let AC denote the ladder and B be the foot of the wall.

Let the length of the ladder AC be x metres.

Given that $\angle CAB = 60^\circ$ and $AB = 3.5$ m.

In the right $\triangle$ CAB, $\cos 60^\circ = \frac{AB}{AC}$

$\Longrightarrow$ $AC = \frac{AB}{\cos 60^\circ}$

$\therefore$ $x = 2 \times 3.5 = 7$ m

Thus, the length of the ladder is 7m.

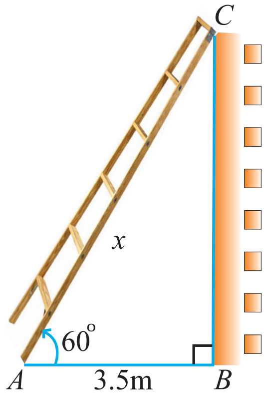


Fig. 7.8

Example 7.16

Find the angular elevation (angle of elevation from the ground level) of the Sun when the length of the shadow of a 30 m long pole is $10\sqrt{3}$ m.

Solution Let S be the position of the Sun and BC be the pole.

Let AB denote the length of the shadow of the pole.

Let the angular elevation of the Sun be θ.

Given that $AB = 10\sqrt{3}$ m and

$BC = 30$ m

In the right $\triangle$ CAB, $\tan\theta = \frac{BC}{AB} = \frac{30}{10\sqrt{3}} = \frac{3}{\sqrt{3}}$

$\Longrightarrow$ $\tan\theta = \sqrt{3}$

$\therefore$ $\theta = 60^\circ$

Thus, the angular elevation of the Sun from the ground level is 60°.

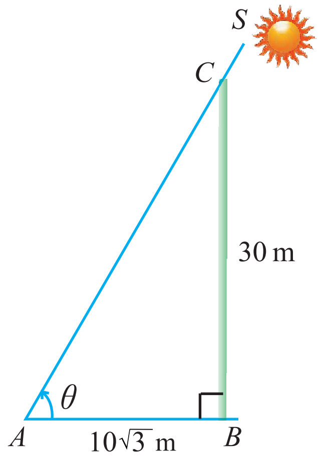


Fig. 7.9

Example 7.17

The angle of elevation of the top of a tower as seen by an observer is 30°. The observer is at a distance of $30\sqrt{3}$ m from the tower. If the eye level of the observer is 1.5 m above the ground level, then find the height of the tower.

Solution Let BD be the height of the tower and AE be the distance of the eye level of the observer from the ground level.

Draw EC parallel to AB such that $AB = EC$.

Given $AB = EC = 30\sqrt{3}$ m and

$AE = BC = 1.5$m

In right angled $\triangle DEC$,

$$\tan 30^\circ = \frac{CD}{EC}$$

$$\Longrightarrow \quad CD = EC \tan 30^\circ = \frac{30\sqrt{3}}{\sqrt{3}}$$

$$\therefore \quad CD = 30\,\text{m}$$

Thus, the height of the tower, $BD = BC + CD$

$$= 1.5 + 30 = 31.5\,\text{m}.$$

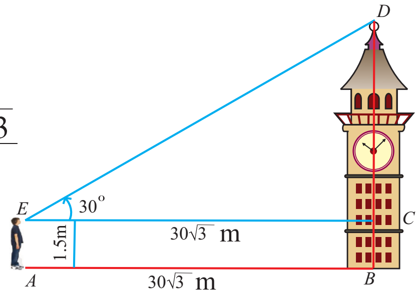


Fig. 7.10

Example 7.18

A vertical tree is broken by the wind. The top of the tree touches the ground and makes an angle 30° with it. If the top of the tree touches the ground 30 m away from its foot, then find the actual height of the tree.

Solution Let C be the point at which the tree is broken and let the top of the tree touch the ground at A.

Let B denote the foot of the tree.

Given $AB = 30$ m and

$\angle CAB = 30^\circ$.

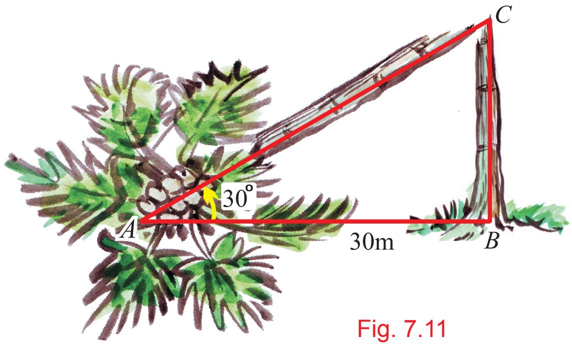


Fig. 7.11

In the right angled $\triangle CAB$,

$$\tan 30^\circ = \frac{BC}{AB}$$

$$\Longrightarrow \quad BC = AB \tan 30^\circ$$

$$\therefore \quad BC = \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3}\,\text{m} \qquad (1)$$

Now, $\cos 30^\circ = \dfrac{AB}{AC}$

$$\Longrightarrow \quad AC = \frac{AB}{\cos 30^\circ}$$

So, $AC = \dfrac{30 \times 2}{\sqrt{3}} = 10\sqrt{3} \times 2 = 20\sqrt{3}\,\text{m}. \qquad (2)$

Thus, the height of the tree $= BC + AC = 10\sqrt{3} + 20\sqrt{3}$

$$= 30\sqrt{3}\,\text{m}.$$

Example 7.19

A jet fighter at a height of 3000 m from the ground, passes directly over another jet fighter at an instance when their angles of elevation from the same observation point are 60° and 45° respectively. Find the distance of the first jet fighter from the second jet at that instant. $(\sqrt{3} = 1.732)$

Solution Let O be the point of observation.

Let A and B be the positions of the two jet fighters at the given instant when one is directly above the other.

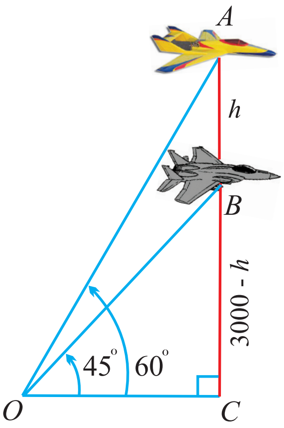


Fig. 7.12

Let C be the point on the ground such that $AC = 3000$ m.

Given $\angle AOC = 60^\circ$ and $\angle BOC = 45^\circ$

Let h denote the distance between the jets at the instant.

In the right angled $\triangle BOC$, $\tan 45^\circ = \dfrac{BC}{OC}$

$$\Longrightarrow \quad OC = BC \qquad (\because \tan 45^\circ = 1)$$

Thus, $$OC = 3000 - h \qquad (1)$$

In the right angled $\triangle AOC$, $\tan 60^\circ = \dfrac{AC}{OC}$

$$\Longrightarrow \quad OC = \frac{AC}{\tan 60^\circ} = \frac{3000}{\sqrt{3}}$$

$$= \frac{3000}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 1000\sqrt{3} \qquad (2)$$

From (1) and (2), we get $3000 - h = 1000\sqrt{3}$

$$\Longrightarrow \quad h = 3000 - 1000 \times 1.732 = 1268 \text{ m}$$

The distance of the first jet fighter from the second jet at that instant is 1268 m.

Example 7.20

The angle of elevation of the top of a hill from the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30°. If the tower is 50 m high, then find the height of the hill.

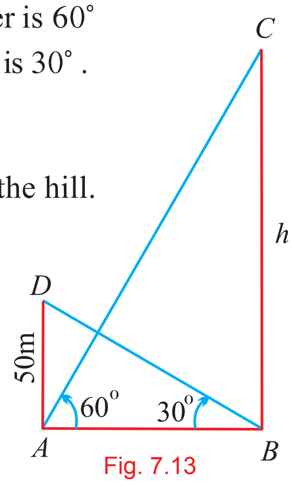


Fig. 7.13

Solution Let AD be the height of tower and BC be the height of the hill.

Given $\angle CAB = 60^\circ$, $\angle ABD = 30^\circ$ and $AD = 50$m.

Let $BC = h$ metres.

Now, in the right angled ΔDAB, $\tan 30^\circ = \dfrac{AD}{AB}$

$$\Longrightarrow \quad AB = \frac{AD}{\tan 30^\circ}$$

$$\therefore \quad AB = 50\sqrt{3} \text{ m} \qquad (1)$$

Also, in the right angled $\triangle CAB$, $\tan 60^\circ = \dfrac{BC}{AB}$

$$\Longrightarrow \quad BC = AB \tan 60^\circ$$

Thus, using (1) we get $h = BC = (50\sqrt{3})\sqrt{3} = 150$ m

Hence, the height of the hill is 150 m.

Example 7.21

A vertical wall and a tower are on the ground. As seen from the top of the tower , the angles of depression of the top and bottom of the wall are 45° and 60° respectively . Find the height of the wall if the height of the tower is 90 m. ($\sqrt{3} = 1.732$)

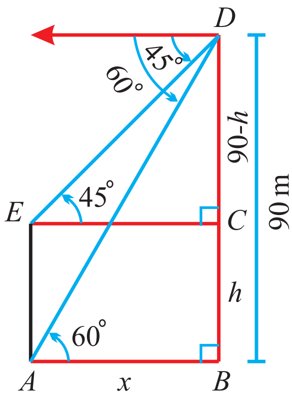


Fig. 7.14

Solution Let AE denote the wall and BD denote the tower.

Draw EC parallel to AB such that $AB=EC$. Thus, $AE = BC$.

Let $AB = x$ metres and $AE = h$ metres.

Given that $BD = 90$ m and $\angle DAB = 60^\circ$, $\angle DEC = 45^\circ$.

Now, $AE = BC = h$ metres

Thus, $CD = BD - BC = 90 - h$.

In the right angled $\triangle DAB$, $\tan 60^\circ = \frac{BD}{AB} = \frac{90}{x}$

$$\Longrightarrow \quad x = \frac{90}{\sqrt{3}} = 30\sqrt{3} \qquad (1)$$

In the right angled $\triangle DEC$, $\tan 45^\circ = \frac{DC}{EC} = \frac{90-h}{x}$

$$\text{Thus,} \quad x = 90 - h \qquad (2)$$

From (1) and (2), we have $90 - h = 30\sqrt{3}$

Thus, the height of the wall, $h = 90 - 30\sqrt{3} = 38.04$ m.

Example 7.22

A girl standing on a lighthouse built on a cliff near the seashore, observes two boats due East of the lighthouse. The angles of depression of the two boats are 30° and 60°. The distance between the boats is 300 m. Find the distance of the top of the lighthouse from the sea level. (Boats and foot of the lighthouse are in a straight line)

Solution Let A and D denote the foot of the cliff and the top of the lighthouse respectively. Let B and C denote the two boats.

Let h metres be the distance of the top of the lighthouse from the sea level.

Let $AB = x$ metres.

Given that $\angle ABD = 60^\circ, \angle ACD = 30^\circ$

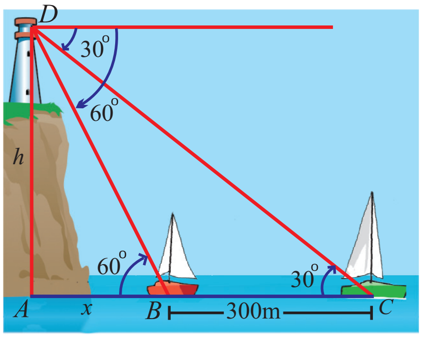


Fig. 7.15

In the right angled ΔABD,

$$\tan 60^\circ = \frac{AD}{AB}$$

$$\Longrightarrow \quad AB = \frac{AD}{\tan 60^\circ}$$

$$\text{Thus,} \quad x = \frac{h}{\sqrt{3}} \qquad (1)$$

Also, in the right angled $\triangle ACD$, we have

$$\tan 30^\circ = \frac{AD}{AC}$$

$$\Longrightarrow \quad AC = \frac{AD}{\tan 30^\circ} \Longrightarrow x + 300 = \frac{h}{\left(\frac{1}{\sqrt{3}}\right)}$$

Thus, $x + 300 = h\sqrt{3}$ (2)

Using (1) in (2), we get $\frac{h}{\sqrt{3}} + 300 = h\sqrt{3}$

$$\Longrightarrow \quad h\sqrt{3} - \frac{h}{\sqrt{3}} = 300$$

$\therefore$ $2h = 300\sqrt{3}$. Thus, $h = 150\sqrt{3}$.

Hence, the height of the lighthouse from the sea level is $150\sqrt{3}$ m.

Example 7.23

A boy spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground level. The distance of his eye level from the ground is 1.2 m. The angle of elevation of the balloon from his eyes at an instant is 60°. After some time, from the same point of observation, the angle of elevation of the balloon reduces to 30°. Find the distance covered by the balloon during the interval.

Solution Let A be the point of observation.

Let E and D be the positions of the balloon when its angles of elevation are 60° and 30° respectively.

Let B and C be the points on the horizontal line such that $BE = CD$.

Let A', B' and C' be the points on the ground such that

$$A'A = B'B = C'C = 1.2\text{ m}.$$

Given that $\angle EAB = 60^\circ$, $\angle DAC = 30^\circ$

$$BB' = CC' = 1.2\text{ m and } C'D = 88.2\text{ m}.$$

Also, we have $BE = CD = 87$ m.

Now, in the right angled $\triangle EAB$, we have

$$\tan 60^\circ = \frac{BE}{AB}$$

Thus, $$AB = \frac{87}{\tan 60^\circ} = \frac{87}{\sqrt{3}} = 29\sqrt{3}$$

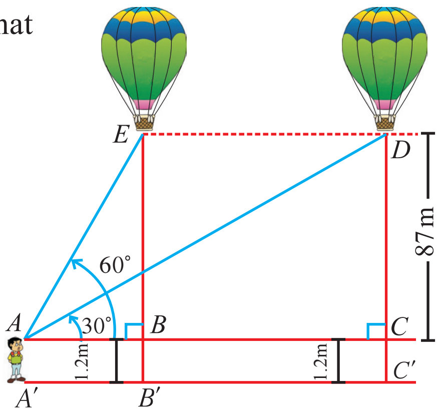


Fig. 7.16

Again in the right angled $\triangle DAC$, we have $\tan 30^\circ = \frac{DC}{AC}$

Thus, $$AC = \frac{87}{\tan 30^\circ} = 87\sqrt{3}.$$

Therefore, the distance covered by the balloon is

$$ED = BC = AC - AB$$

$$= 87\sqrt{3} - 29\sqrt{3} = 58\sqrt{3}\text{ m}.$$

Example 7.24

A flag post stands on the top of a building. From a point on the ground, the angles of elevation of the top and bottom of the flag post are 60° and 45° respectively. If the height of the flag post is $10\,\text{m}$, find the height of the building. $(\sqrt{3} = 1.732)$

Solution

Let A be the point of observation and B be the foot of the building.

Let BC denote the height of the building and CD denote height of the flag post.

Given that $\angle CAB = 45^\circ$, $\angle DAB = 60^\circ$ and $CD = 10\,\text{m}$

Let $BC = h$ metres and $AB = x$ metres.

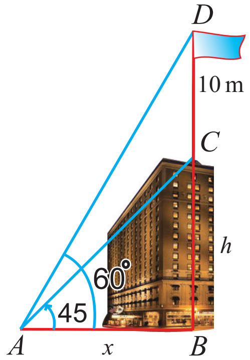


Fig. 7.17

Now, in the right angled $\triangle CAB$,

$$\tan 45^\circ = \frac{BC}{AB}.$$

Thus, $AB = BC$ i.e., $x = h$ (1)

Also, in the right angled $\triangle DAB$,

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\implies AB = \frac{h+10}{\tan 60^\circ} \implies x = \frac{h+10}{\sqrt{3}} \qquad (2)$$

From (1) and (2), we get $h = \dfrac{h+10}{\sqrt{3}}$

$$\implies \sqrt{3}\,h - h = 10$$

$$\implies h = \left(\frac{10}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \frac{10(\sqrt{3}+1)}{3-1}$$

$$= 5\,(2.732) = 13.66\,\text{m}$$

Hence, the height of the building is $13.66\,\text{m}$.

Example 7.25

A man on the deck of a ship, $14\,\text{m}$ above the water level, observes that the angle of elevation of the top of a cliff is 60° and the angle of depression of the base of the cliff is 30°. Find the height of the cliff.

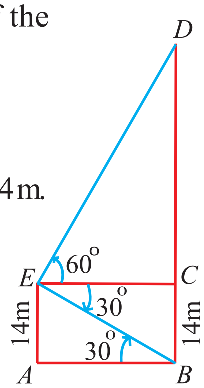


Fig. 7.18

Solution Let BD be the height of the cliff.

Let A be the position of ship and E be the point of observation so that $AE = 14\,\text{m}$.

Draw EC parallel to AB such that $AB = EC$.

Given that $\angle ABE = 30^\circ$, $\angle DEC = 60^\circ$

In the right angled $\triangle ABE$, $\tan 30^\circ = \dfrac{AE}{AB}$

$$\therefore \qquad AB = \frac{AE}{\tan 30^\circ} \quad \Longrightarrow AB = 14\sqrt{3}$$

Thus, $\qquad EC = 14\sqrt{3} \qquad\qquad (\because\ AB = EC)$

In the right angled $\triangle DEC$, $\quad \tan 60^\circ = \dfrac{CD}{EC}$

$$\therefore\ CD = EC\tan 60^\circ \Longrightarrow CD = (14\sqrt{3})\sqrt{3} = 42\text{ m}$$

Thus, the height of the cliff, $BD = BC + CD = 14 + 42 = 56$ m.

Example 7.26

The angle of elevation of an aeroplane from a point A on the ground is 60°. After a flight of 15 seconds horizontally, the angle of elevation changes to 30°. If the aeroplane is flying at a speed of 200 m/s, then find the constant height at which the aeroplane is flying.

Solution Let A be the point of observation.

Let E and D be positions of the aeroplane initially and after 15 seconds respectively.

Let BE and CD denote the constant height at which the aeroplane is flying.

Given that $\angle DAC = 30^\circ$, $\angle EAB = 60^\circ$.

Let $BE = CD = h$ metres.

Let $AB = x$ metres.

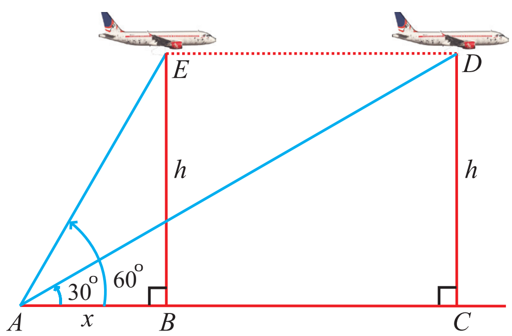


Fig. 7.19

The distance covered in 15 seconds,

$$ED = 200 \times 15 = 3000\text{ m} \qquad \text{(distance travelled = speed} \times \text{time)}$$

Thus, $\quad BC = 3000$ m.

In the right angled $\triangle DAC$,

$$\tan 30^\circ = \frac{CD}{AC}$$

$$\Longrightarrow \qquad CD = AC\tan 30^\circ$$

Thus, $\qquad h = (x + 3000)\dfrac{1}{\sqrt{3}}. \qquad\qquad (1)$

In the right angled $\triangle EAB$,

$$\tan 60^\circ = \frac{BE}{AB}$$

$$\Longrightarrow \qquad BE = AB\tan 60^\circ \Longrightarrow h = \sqrt{3}\,x \qquad\qquad (2)$$

From (1) and (2), we have $\quad \sqrt{3}\,x = \dfrac{1}{\sqrt{3}}(x + 3000)$

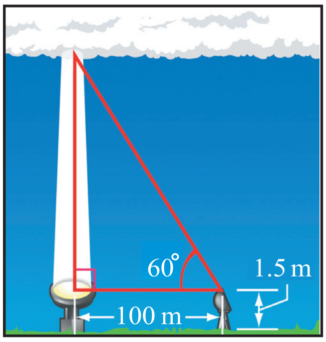
$$\Longrightarrow \qquad 3x = x + 3000 \qquad \Longrightarrow \qquad x = 1500\text{ m.}$$

Thus, from (2) it follows that $h = 1500\sqrt{3}$ m.

The constant height at which the aeroplane is flying, is $1500\sqrt{3}$ m.

Exercise 7.2

1. A ramp for unloading a moving truck, has an angle of elevation of 30°. If the top of the ramp is $0.9\,m$ above the ground level, then find the length of the ramp.
2. A girl of height $150\,cm$ stands in front of a lamp-post and casts a shadow of length $150\sqrt{3}$ cm on the ground. Find the angle of elevation of the top of the lamp-post .
3. Suppose two insects A and B can hear each other up to a range of $2\,m$.. The insect A is on the ground $1\,m$ away from a wall and sees her friend B on the wall, about to be eaten by a spider. If A sounds a warning to B and if the angle of elevation of B from A is 30°, will the spider have a meal or not ? (Assume that B escapes if she hears A calling)
4. To find the cloud ceiling, one night an observer directed a spotlight vertically at the clouds. Using a theodolite placed 100 m from the spotlight and $1.5\,m$ above the ground, he found the angle of elevation to be 60°. How high was the cloud ceiling? (Hint : See figure)



(**Note:** Cloud ceiling is the lowest altitude at which solid cloud is present. The cloud ceiling at airports must be sufficiently high for safe take offs and landings. At night the cloud ceiling can be determined by illuminating the base of the clouds by a spotlight pointing vertically upward.)

5. A simple pendulum of length $40\,cm$ subtends 60° at the vertex in one full oscillation. What will be the shortest distance between the initial position and the final position of the bob? (between the extreme ends)
6. Two crows A and B are sitting at a height of $15\,m$ and $10\,m$ in two different trees vertically opposite to each other . They view a vadai (an eatable) on the ground at an angle of depression 45° and 60°respectively. They start at the same time and fly at the same speed along the shortest path to pick up the vadai. Which bird will succeed in it? Hint : (foot of two trees and vadai (an eatable) are in a straight line)
7. A lamp-post stands at the centre of a circular park. Let P and Q be two points on the boundary such that PQ subtends an angle 90° at the foot of the lamp-post and the angle of elevation of the top of the lamp post from P is 30°. If $PQ = 30$ m, then find the height of the lamp post.
8. A person in an helicopter flying at a height of 700 m, observes two objects lying opposite to each other on either bank of a river. The angles of depression of the objects are 30° and 45°. Find the width of the river. ($\sqrt{3} = 1.732$)
9. A person X standing on a horizontal plane, observes a bird flying at a distance of 100 m from him at an angle of elevation of 30°. Another person Y standing on the roof of a 20 m high building, observes the bird at the same time at an angle of elevation of 45°. If X and Y are on the opposite sides of the bird, then find the distance of the bird from Y.

10. A student sitting in a classroom sees a picture on the black board at a height of 1.5 m from the horizontal level of sight. The angle of elevation of the picture is 30°. As the picture is not clear to him, he moves straight towards the black board and sees the picture at an angle of elevation of 45°. Find the distance moved by the student.

11. A boy is standing at some distance from a 30 m tall building and his eye level from the ground is 1.5 m. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

12. From the top of a lighthouse of height 200 feet, the lighthouse keeper observes a Yacht and a Barge along the same line of sight . The angles of depression for the Yacht and the Barge are 45° and 30° respectively. For safety purposes the two sea vessels should be atleast 300 feet apart. If they are less than 300 feet , the keeper has to sound the alarm. Does the keeper have to sound the alarm ?

13. A boy standing on the ground, spots a balloon moving with the wind in a horizontal line at a constant height . The angle of elevation of the balloon from the boy at an instant is 60°. After 2 minutes, from the same point of observation,the angle of elevation reduces to 30°. If the speed of wind is $29\sqrt{3}$ m/min. then, find the height of the balloon from the ground level.

14. A straight highway leads to the foot of a tower . A man standing on the top of the tower spots a van at an angle of depression of 30°. The van is approaching the tower with a uniform speed. After 6 minutes, the angle of depression of the van is found to be 60°. How many more minutes will it take for the van to reach the tower?

15. The angles of elevation of an artificial earth satellite is measured from two earth stations, situated on the same side of the satellite, are found to be 30° and 60°. The two earth stations and the satellite are in the same vertical plane. If the distance between the earth stations is 4000 km, find the distance between the satellite and earth. ($\sqrt{3} = 1.732$)

16. From the top of a tower of height 60 m, the angles of depression of the top and the bottom of a building are observed to be 30° and 60° respectively. Find the height of the building.

17. From the top and foot of a 40 m high tower, the angles of elevation of the top of a lighthouse are found to be 30° and 60° respectively. Find the height of the lighthouse. Also find the distance of the top of the lighthouse from the foot of the tower.

18. The angle of elevation of a hovering helicopter as seen from a point 45 m above a lake is 30° and the angle of depression of its reflection in the lake, as seen from the same point and at the same time, is 60°. Find the distance of the helicopter from the surface of the lake.

Exercise 7.3

Choose the correct answer

1. $(1-\sin^2\theta)\sec^2\theta =$

(A) 0 (B) 1 (C) $\tan^2\theta$ (D) $\cos^2\theta$

2. $(1+\tan^2\theta)\sin^2\theta =$

(A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $\cot^2\theta$

3. $(1-\cos^2\theta)(1+\cot^2\theta) =$

(A) $\sin^2\theta$ (B) 0 (C) 1 (D) $\tan^2\theta$

4. $\sin(90^\circ-\theta)\cos\theta + \cos(90^\circ-\theta)\sin\theta =$

(A) 1 (B) 0 (C) 2 (D) –1

5. $1-\dfrac{\sin^2\theta}{1+\cos\theta} =$

(A) $\cos\theta$ (B) $\tan\theta$ (C) $\cot\theta$ (D) $\operatorname{cosec}\theta$

6. $\cos^4 x - \sin^4 x =$

(A) $2\sin^2 x - 1$ (B) $2\cos^2 x - 1$ (C) $1+2\sin^2 x$ (D) $1-2\cos^2 x$.

7. If $\tan\theta = \dfrac{a}{x}$, then the value of $\dfrac{x}{\sqrt{a^2+x^2}} =$

(A) $\cos\theta$ (B) $\sin\theta$ (C) $\operatorname{cosec}\theta$ (D) $\sec\theta$

8. If $x = a\sec\theta,\ y = b\tan\theta$, then the value of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} =$

(A) 1 (B) –1 (C) $\tan^2\theta$ (D) $\operatorname{cosec}^2\theta$

9. $\dfrac{\sec\theta}{\cot\theta+\tan\theta} =$

(A) $\cot\theta$ (B) $\tan\theta$ (C) $\sin\theta$ (D) $-\cot\theta$

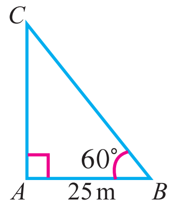
10. $\dfrac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta} + \dfrac{\cos(90^\circ-\theta)\cos\theta}{\cot\theta} =$

(A) $\tan\theta$ (B) 1 (C) –1 (D) $\sin\theta$

11. In the adjoining figure, $AC =$

(A) 25 m (B) $25\sqrt{3}$ m

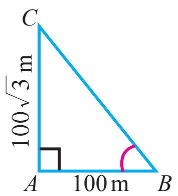
(C) $\dfrac{25}{\sqrt{3}}$ m (D) $25\sqrt{2}$ m



12. In the adjoining figure $\angle ABC =$

(A) 45° (B) 30°

(C) 60° (D) 50°



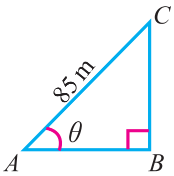
13. A man is 28.5 m away from a tower. His eye level above the ground is 1.5 m. The angle of elevation of the tower from his eyes is 45°. Then the height of the tower is

(A) 30 m (B) 27.5 m (C) 28.5 m (D) 27 m

14. In the adjoining figure, $\sin\theta = \frac{15}{17}$. Then $BC =$

(A) 85 m (B) 65 m

(C) 95 m (D) 75 m



15. $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) =$

(A) $\cos^2\theta - \sin^2\theta$ (B) $\sin^2\theta - \cos^2\theta$

(C) $\sin^2\theta + \cos^2\theta$ (D) 0

16. $(1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta) =$

(A) $\tan^2\theta - \sec^2\theta$ (B) $\sin^2\theta - \cos^2\theta$

(C) $\sec^2\theta - \tan^2\theta$ (D) $\cos^2\theta - \sin^2\theta$

17. $(\cos^2\theta - 1)(\cot^2\theta + 1) + 1 =$

(A) 1 (B) –1 (C) 2 (D) 0

18. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} =$

(A) $\cos^2\theta$ (B) $\tan^2\theta$ (C) $\sin^2\theta$ (D) $\cot^2\theta$

19. $\sin^2\theta + \dfrac{1}{1+\tan^2\theta} =$

(A) $\operatorname{cosec}^2\theta + \cot^2\theta$ (B) $\operatorname{cosec}^2\theta - \cot^2\theta$

(C) $\cot^2\theta - \operatorname{cosec}^2\theta$ (D) $\sin^2\theta - \cos^2\theta$

20. $9\tan^2\theta - 9\sec^2\theta =$

(A) 1 (B) 0 (C) 9 (D) –9

Do you know?

Paul Erdos (26th March, 1913 – 20th September, 1996) was a Hungarian Mathematician. Erdos was one of the most prolific publishers of research articles in mathematical history, comparable only with Leonhard Euler. He wrote around 1,475 mathematical articles in his life lifetime, while Euler credited with approximately 800 research articles. He strongly believed in and practised mathematics as a social activity, having 511 different collaborators in his lifetime.

8

MENSURATION

Measure what is measurable, and make measurable what is not so

-Galileo Galilei

- Introduction
- Surface area and volume
  - Cylinder
  - Cone
  - Sphere
- Combined figures and invariant volumes

Archimedes

(287 BC - 212 BC)

Greece

Archimedes is remembered as the greatest mathematician of the ancient era.

He contributed significantly in geometry regarding the areas of plane figures and the areas as well as volumes of curved surfaces.

8.1 Introduction

The part of geometry which deals with measurement of lengths of lines, perimeters and areas of plane figures and surface areas and volumes of solid objects is called "Mensuration". The study of measurement of objects is essential because of its uses in many aspects of every day life. In elementary geometry one considers plane, multifaced surfaces as well as certain curved surfaces of solids (for example spheres).

"Surface Area to Volume" ratio has been widely acknowledged as one of the big ideas of Nanoscience as it lays the foundation for understanding size dependent properties that characterise Nanoscience scale and technology.

In this chapter, we shall learn how to find surface areas and volumes of solid objects such as cylinder, cone, sphere and combined objects

8.2 Surface Area

Archimedes of Syracuse, Sicily was a Greek Mathematician who proved that of the volume of a sphere is equal to two-thirds the volume of a circumscribed cylinder. He regarded this as his most vital achievement. He used the method of exhaustion to calculate the area under the arc of a parabola.

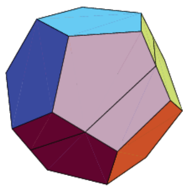


Fig. 8.1

Surface area is the measurement of exposed area of a solid object. Thus, the surface area is the area of all outside surfaces of a 3-dimensional object. The adjoining figures illustrate surface areas of some solids.

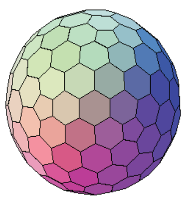


Fig. 8.2

8.2.1 Right Circular Cylinder

If we take a number of circular sheets of paper or cardboard of the same shape and size and stack them up in a vertical pile, then by this process, we shall obtain a solid object known as a Right Circular Cylinder. Note that it has been kept at right angles to the base, and the base is circular. (See Fig. 8.3)

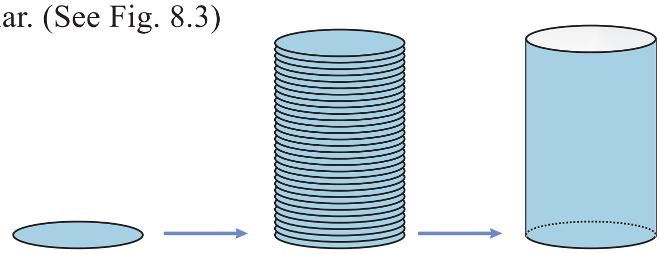


Fig. 8.3

Definition

If a rectangle revolves about its one side and completes a full rotation, the solid thus formed is called a right circular cylinder.

Activity

Let $ABCD$ be a rectangle. Assume that it revolves about its side AB and completes a full rotation. This revolution generates a right circular cylinder as shown in the figures. AB is called the axis of the cylinder. The length AB is the length or the height of the cylinder and AD or BC is called its radius.

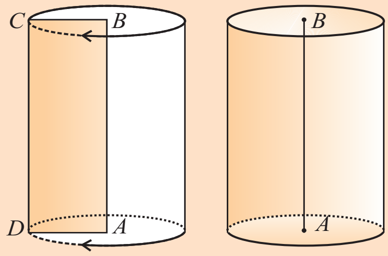


Fig. 8.4

Note

(i) If the base of a cylinder is not circular then it is called oblique cylinder.

(ii) If the base is circular but not perpendicular to the axis of the cylinder, then the cylinder is called circular cylinder.

(iii) If the axis is perpendicular to the circular base, then the cylinder is called right circular cylinder.

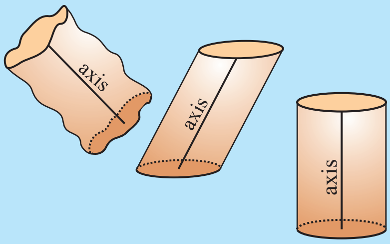


Fig. 8.5

(i) Curved Surface area of a solid right circular cylinder

In the adjoining figure, the bottom and top face of the right circular cylinder are concurrent circular regions, parallel to each other. The vertical surface of the cylinder is curved and hence its area is called the curved surface or lateral surface area of the cylinder.

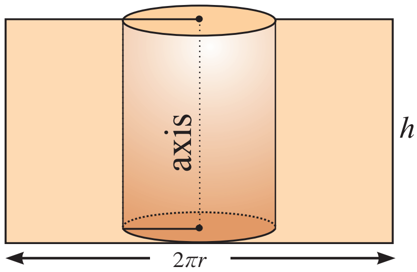


Fig. 8.6

Curved Surface Area of a cylinder, CSA = Circumference of the base $\times$ Height $= 2\pi r \times h$

$$= 2\pi rh \text{ sq. units.}$$

(ii) Total Surface Area of a solid right circular cylinder

Total Surface Area, TSA $= $ Area of the Curved Surface Area $+ 2\times$ Base Area

$$= 2\pi rh + 2\times \pi r^2$$

Thus, TSA $= 2\pi r(h+r)$ sq.units.

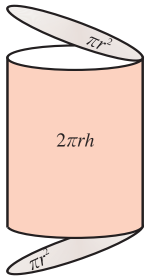


Fig. 8.7

(iii) Right circular hollow cylinder

Solids like iron pipe, rubber tube, etc., are in the shape of hollow cylinders. For a hollow cylinder of height h with external and internal radii R and r respectively,

we have, curved surface area, CSA $=$ External surface area $+$ Internal surface area

$$= 2\pi Rh + 2\pi rh$$

Thus, CSA $= 2\pi h(R+r)$ sq.units

Total surface area, TSA $= $ CSA $+ 2\times$ Base area

$$= 2\pi h(R+r) + 2\times[\pi R^2 - \pi r^2]$$

$$= 2\pi h(R+r) + 2\pi(R+r)(R-r)$$

$\therefore$ TSA $= 2\pi(R+r)(R-r+h)$ sq.units.

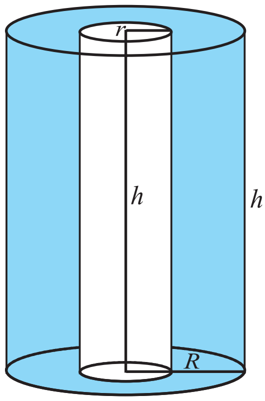


Fig. 8.8

Remark

Thickness of the hollow cylinder, w $= R - r$.

> **Note**
> In this chapter, for π we take an approximate value $\frac{22}{7}$ whenever it is required.

Example 8.1

A solid right circular cylinder has radius 7cm and height 20cm. Find its (i) curved surface area and (ii) total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

Given that $r = 7$cm and $h = 20$cm

Curved surface area, CSA $= 2\pi rh$

$$= 2\times\frac{22}{7}\times 7\times 20$$

Thus, the curved surface area $= 880$ sq.cm

Now, the total surface area $= 2\pi r(h+r)$

$$= 2\times\frac{22}{7}\times 7\times[20+7] = 44\times 27$$

Thus, the total surface area $= 1188$ sq.cm.

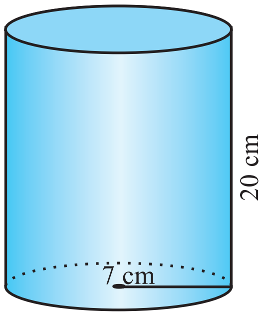


Fig. 8.9

Example 8.2

If the total surface area of a solid right circular cylinder is 880 sq.cm and its radius is 10 cm, find its curved surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

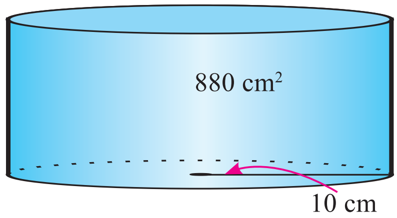


Fig. 8.10

Let S be the total surface area of the solid right circular cylinder.

Given that $r = 10$ cm and $S = 880$ cm^2

Now, $S = 880 \implies 2\pi r[h + r] = 880$

$$\implies \quad 2 \times \frac{22}{7} \times 10[h + 10] = 880$$

$$\implies \quad h + 10 = \frac{880 \times 7}{2 \times 22 \times 10}$$

$$\implies \quad h + 10 = 14$$

Thus, the height of the cylinder, $h = 4$ cm

Now, the curved surface area, CSA is

$$2\pi rh = 2 \times \frac{22}{7} \times 10 \times 4 = \frac{1760}{7}$$

Aliter :

$$\text{CSA} = \text{TSA} - 2\times \text{ Area of the base}$$

$$= 880 - 2 \times \pi r^2$$

$$= 880 - 2 \times \frac{22}{7} \times 10^2$$

$$= \frac{1760}{7} = 251\frac{3}{7} \text{ sq.cm.}$$

Thus, the curved surface area of the cylinder = $251\frac{3}{7}$ sq.cm.

Example 8.3

The ratio between the base radius and the height of a solid right circular cylinder is 2 : 5. If its curved surface area is $\frac{3960}{7}$ sq.cm, find the height and radius. (use $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $r : h = 2 : 5 \implies \frac{r}{h} = \frac{2}{5}$. Thus, $r = \frac{2}{5}h$

Now, the curved surface area, CSA = $2\pi rh$

$$\implies \quad 2 \times \frac{22}{7} \times \frac{2}{5} \times h \times h = \frac{3960}{7}$$

$$\implies \quad h^2 = \frac{3960 \times 7 \times 5}{2 \times 22 \times 2 \times 7} = 225$$

Thus, $h = 15 \implies r = \frac{2}{5}h = 6.$

Hence, the height of the cylinder is 15 cm and the radius is 6 cm.

Example 8.4

The diameter of a road roller of length 120 cm is 84 cm. If it takes 500 complete revolutions to level a playground, then find the cost of levelling it at the cost of 75 paise per square metre. (Take $\pi = \frac{22}{7}$)

Solution Given that $r = 42$ cm, $h = 120$ cm

$$\left.\begin{array}{r}\text{Area covered by the roller}\\ \text{in one revolution}\end{array}\right\} = \left\{\begin{array}{l}\text{Curved Surface Area}\\ \text{of the road roller.}\end{array}\right.$$

$$= 2\pi rh$$

$$= 2 \times \frac{22}{7} \times 42 \times 120$$

$$= 31680 \text{ cm}^2.$$

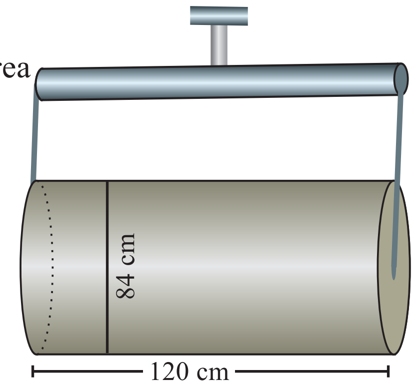


Fig. 8.11

$$\left.\begin{array}{r}\text{Area covered by the}\\ \text{roller in 500 revolutions}\end{array}\right\} = 31680 \times 500$$

$$= 15840000 \text{ cm}^2$$

$$= \frac{15840000}{10000} = 1584 \text{ m}^2 \qquad (10{,}000 \text{ cm}^2 = 1 \text{ sq.m})$$

Cost of levelling per 1sq.m. $= ₹\,\frac{75}{100}$

Thus, cost of levelling the play ground $= \frac{1584 \times 75}{100} = ₹\,1188$.

Example 8.5

The internal and external radii of a hollow cylinder are 12 cm and 18 cm respectively. If its height is 14cm, then find its curved surface area and total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the internal and external radii and the height of a hollow cylinder respectively.

Given that $r = 12$ cm, $R = 18$ cm, $h = 14$ cm

Now, curved surface area, CSA $= 2\pi h(R+r)$

$$\text{Thus, CSA} = 2 \times \frac{22}{7} \times 14 \times (18 + 12)$$

$$= 2640 \text{ sq.cm}$$

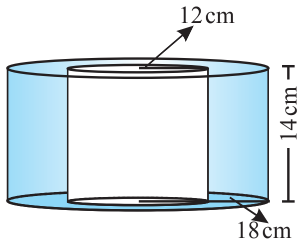


Fig. 8.12

$$\text{Total surface area,}\quad \text{TSA} = 2\pi(R + r)(R - r + h)$$

$$= 2 \times \frac{22}{7} \times (18 + 12)(18 - 12 + 14)$$

$$= 2 \times \frac{22}{7} \times 30 \times 20 = \frac{26400}{7}.$$

Thus, the total surface area $= 3771\frac{3}{7}$ sq.cm.

8.2.2 Right Circular Cone

In our daily life we come across many solids or objects like ice cream container, the top of the temple car, the cap of a clown in a circus, the mehandi container. Mostly the objects mentioned above are in the shape of a right circular cone.

A cone is a solid object that tapers smoothly from a flat base to a point called vertex. In general, the base may not be of circular shape. Here, cones are assumed to be right circular, where right means that the axis that passes through the centre of the base is at right angles to its plane, and circular means that the base is a circle. In this section, let us define a right circular cone and find its surface area. One can visualise a cone through the following activity.

Activity

Take a thick paper and cut a right angled $\triangle ABC$, right angled at B. Paste a long thick string along one of the perpendicular sides say AB of the triangle. Hold the string with your hands on either side of the triangle and rotate the triangle about the string.

What happens? Can you recognize the shape formed on the rotation of the triangle around the string?. The shape so formed is a right circular cone.

If a right angled $\triangle ABC$ is revolved 360° about the side AB containing the right angle, the solid thus formed is called a right circular cone.

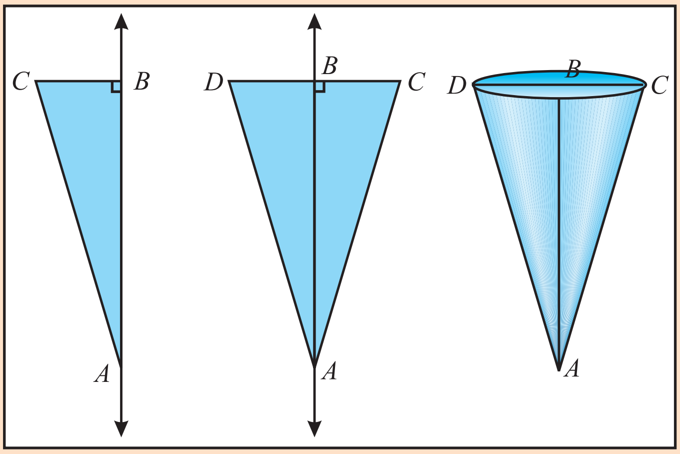


Fig. 8.13

The length AB is called the height of the cone.

The length BC is called the radius of its base ($BC = r$). The length AC is called the slant height l of the cone ($AC = AD = l$).

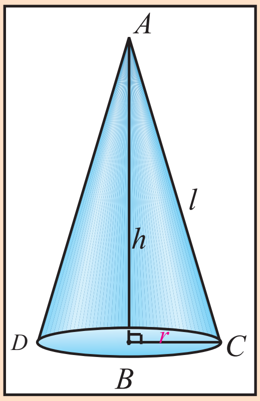


Fig. 8.14

In the right angled $\triangle ABC$

We have, $l = \sqrt{h^2 + r^2}$ (Pythagoras theorem)

$$h = \sqrt{l^2 - r^2}$$

$$r = \sqrt{l^2 - h^2}$$

Note

(i) If the base of a cone is not circular then, it is called **oblique cone.**

(ii) If the circular base is not perpendicular to the axis then, it is called **circular cone.**

(iii) If the vertex is directly above the centre of the circular base then, it is a **right circular cone.**

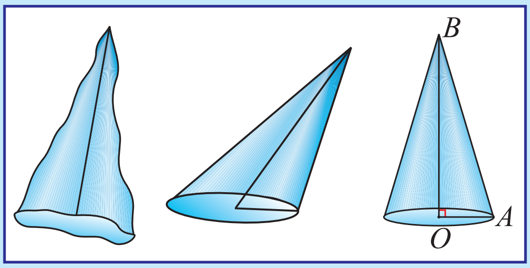


Fig. 8.15

(i) Curved surface area of a hollow cone

Let us consider a sector with radius l and central angle $\theta°$. Let L denote the length of the arc. Thus, $\frac{2\pi l}{L} = \frac{360°}{\theta°}$

$$\implies \quad L = 2\pi l \times \frac{\theta°}{360°} \qquad (1)$$

Now, join the radii of the sector to obtain a right circular cone.

Let r be the radius of the cone.

Hence, $\quad L = 2\pi r$

From (1) we obtain,

$$2\pi r = 2\pi l \times \frac{\theta°}{360°}$$

$$\implies \quad r = l\left(\frac{\theta°}{360°}\right)$$

$$\implies \quad \frac{r}{l} = \left(\frac{\theta°}{360°}\right)$$

Let A be the area of the sector. Then

$$\frac{\pi l^2}{A} = \frac{360°}{\theta°} \qquad (2)$$

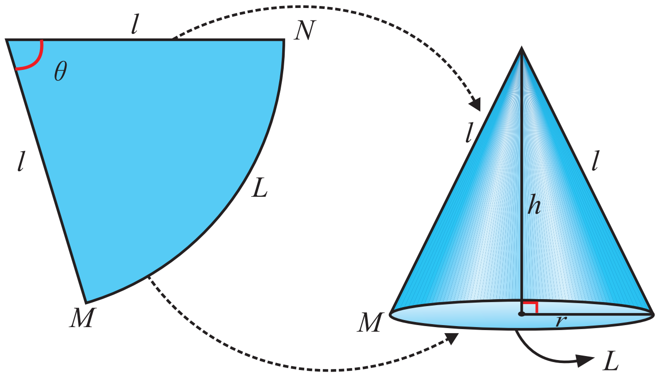


Fig. 8.16

Remarks

When a sector of a circle is folded into a cone, the following conversions are taking place:

Sector	Cone
Radius (l) →	Slant height (l)
Arc Length (L) →	Perimeter of the base $2\pi r$
Area →	Curved Surface Area πrl

Then the curved surface area of the cone $\Big\}$ = Area of the sector

Thus, the area of the curved surface of the cone $\Big\}$ $A = \pi l^2\left(\frac{\theta°}{360°}\right) = \pi l^2\left(\frac{r}{l}\right)$.

Hence, the curved surface area of the cone $= \pi rl$ sq.units.

(ii) Total surface area of the solid right circular cone

Total surface area of the solid cone = $\Big\{$ Curved surface area of the cone + Area of the base

$$= \pi rl + \pi r^2$$

Total surface area of the solid cone $= \pi r(l + r)$ sq.units.

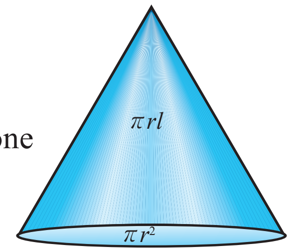


Fig. 8.17

Example 8.6

Radius and slant height of a solid right circular cone are 35cm and 37cm respectively. Find the curved surface area and total surface area of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r and l be the radius and the slant height of the solid right circular cone respectively.

$$r = 35 \text{ cm}, \; l = 37 \text{ cm}$$

Curved surface area, $\text{CSA} = \pi r l = \pi(35)(37)$

$\text{CSA} = 4070$ sq.cm

Total surface area, $\text{TSA} = \pi r[l + r]$

$$= \frac{22}{7} \times 35 \times [37 + 35]$$

Thus, $\text{TSA} = 7920$ sq.cm.

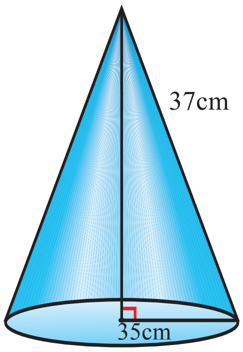


Fig. 8.18

Example 8.7

Let O and C be the centre of the base and the vertex of a right circular cone. Let B be any point on the circumference of the base. If the radius of the cone is 6 cm and if $\angle OBC = 60^o$, then find the height and curved surface area of the cone.

Solution Given that radius OB = 6 cm and $\angle OBC = 60^o$.

In the right angled $\triangle OBC$,

$$\cos 60^o = \frac{OB}{BC}$$

$$\Longrightarrow \quad BC = \frac{OB}{\cos 60^\circ}$$

$$\therefore \quad BC = \frac{6}{(\frac{1}{2})} = 12 \text{ cm}$$

Thus, the slant height of the cone, $l = 12$ cm

In the right angled $\triangle OBC$, we have

$$\tan 60^o = \frac{OC}{OB}$$

$$\Longrightarrow \quad OC = OB \tan 60^\circ = 6\sqrt{3}$$

Thus, the height of the cone, OC = $6\sqrt{3}$ cm

Now, the curved surface area is $\pi r l = \pi \times 6 \times 12 = 72\pi \text{ cm}^2$.

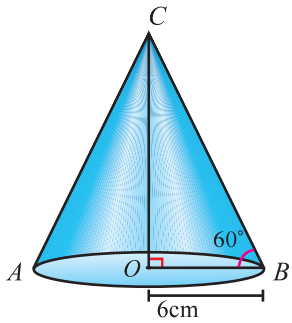


Fig. 8.19

Example 8.8

A sector containing an angle of 120° is cut off from a circle of radius 21 cm and folded into a cone. Find the curved surface area of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r be the base radius of the cone.

Angle of the sector, $\theta = 120^\circ$

Radius of the sector, $R = 21$ cm

When the sector is folded into a right circular cone, we have

circumference of the base of the cone

$$= \text{Length of the arc}$$

$$\Longrightarrow \quad 2\pi r = \frac{\theta}{360^\circ} \times 2\pi R$$

$$\Longrightarrow \quad r = \frac{\theta}{360^\circ} \times R$$

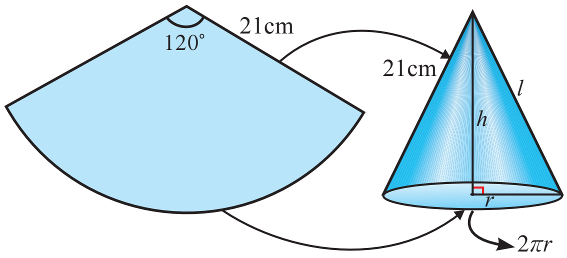


Fig. 8.20

Thus, the base radius of the cone, $r = \frac{120^\circ}{360^\circ} \times 21 = 7$ cm.

Also, the slant height of the cone ,

$$l = \text{Radius of the sector}$$

Thus, $\quad l = R \Longrightarrow l = 21$ cm.

Now , the curved surface area of the cone,

$$\text{CSA} = \pi r l$$

$$= \frac{22}{7} \times 7 \times 21 = 462.$$

Thus, the curved surface area of the cone is 462 sq.cm.

Aliter :

CSA of the cone = Area of the sector

$$= \frac{\theta^\circ}{360^\circ} \times \pi \times R^2$$

$$= \frac{120}{360} \times \frac{22}{7} \times 21 \times 21$$

$$= 462 \text{ sq.cm.}$$

8.2.3 Sphere

If a circular disc is rotated about one of its diameter, the solid thus generated is called sphere. Thus sphere is a 3- dimensional object which has surface area and volume.

(i) Curved surface area of a solid sphere

Activity

Take a circular disc, paste a string along a diameter of the disc and rotate it 360°. The object so created looks like a ball. The new solid is called sphere.

The following activity may help us to visualise the surface area of a sphere as four times the area of the circle with the same radius.

- Take a plastic ball.
- Fix a pin at the top of the ball.
- Wind a uniform thread over the ball so as to cover the whole curved surface area.
- Unwind the thread and measure the length of the thread used.
- Cut the thread into four equal parts.
- Place the strings as shown in the figures.
- Measure the radius of the sphere and the circles formed.

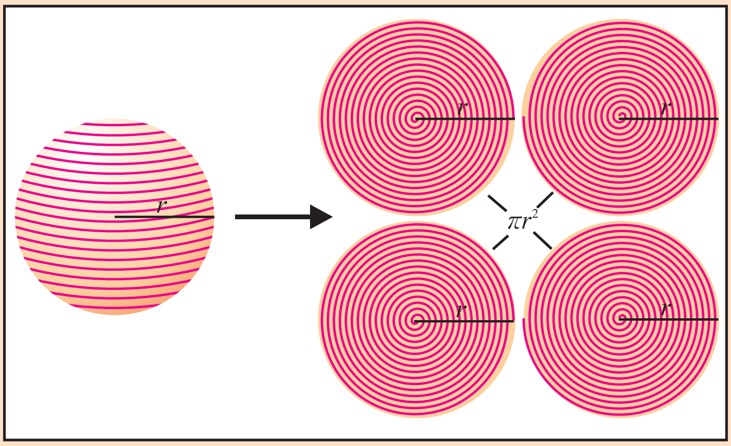


Fig. 8.21

Now, the radius of the sphere = radius of the four equal circles.

Thus, curved surface area of the sphere, CSA = $4 \times$ Area of the circle $= 4 \times \pi r^2$

$\therefore$ The curved surface area of a sphere = $4\pi r^2$ sq. units.

(ii) Solid hemisphere

A plane passing through the centre of a solid sphere divides the sphere into two equal parts. Each part of the sphere is called a solid hemisphere.

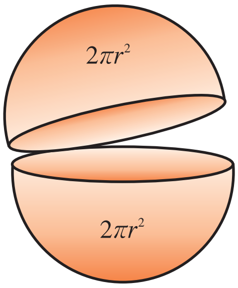


Fig. 8.22

Curved surface area of a hemisphere $= \dfrac{\text{CSA of the Sphere}}{2}$

$$= \frac{4\pi r^2}{2} = 2\pi r^2 \text{ sq.units.}$$

Total surface area of a hemisphere, TSA = Curved Surface Area + Area of the base Circle

$$= 2\pi r^2 + \pi r^2$$
$$= 3\pi r^2 \text{ sq.units.}$$

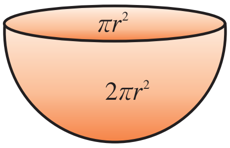


Fig. 8.23

(iii) Hollow hemisphere

Let R and r be the outer and inner radii of the hollow hemisphere.

Now, its curved surface area = Outer surface area + Inner surface area

$$= 2\pi R^2 + 2\pi r^2$$
$$= 2\pi(R^2 + r^2) \text{ sq.units.}$$

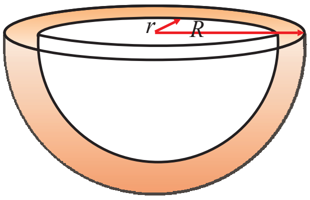


Fig. 8.24

The total surface area $= \begin{cases} \text{Outer surface area + Inner surface area} \\ + \text{ Area at the base} \end{cases}$

$$= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$$
$$= 2\pi(R^2 + r^2) + \pi(R+r)(R-r) \text{ sq.units.}$$
$$= \pi(3R^2 + r^2) \text{ sq. units}$$

Example 8.9

A hollow sphere in which a circus motorcyclist performs his stunts, has an inner diameter of 7 m. Find the area available to the motorcyclist for riding. (Take $\pi = \frac{22}{7}$)

Solution Inner diameter of the hollow sphere, $2r = 7$ m.

Available area to the motorcyclist for riding = Inner surface area of the sphere

$$= 4\pi r^2 = \pi(2r)^2$$
$$= \frac{22}{7} \times 7^2$$

Available area to the motorcyclist for riding = 154 sq.m.

Example 8.10

Total surface area of a solid hemisphere is 675π sq.cm. Find the curved surface area of the solid hemisphere.

Solution Given that the total surface area of the solid hemisphere,

$$3\pi r^2 = 675\pi \text{ sq. cm}$$

$$\Longrightarrow \quad r^2 = 225$$

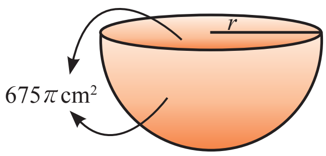


Fig. 8.25

Now, the curved surface area of the solid hemisphere,

$$\text{CSA} = 2\pi r^2 = 2\pi \times 225 = 450\pi \text{ sq.cm.}$$

Example 8.11

The thickness of a hemispherical bowl is 0.25 cm. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.(Take $\pi = \frac{22}{7}$)

Solution Let r, R and w be the inner and outer radii and thickness of the hemispherical bowl respectively.

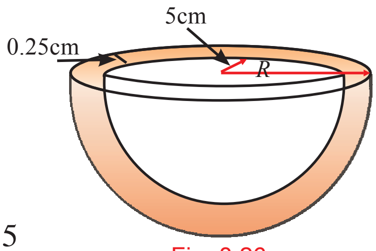


Fig. 8.26

Given that $r = 5\text{ cm}, w = 0.25\text{ cm}$

$\therefore \quad R = r + w = 5 + 0.25 = 5.25\text{ cm}$

Now, outer surface area of the bowl $= 2\pi R^2$

$$= 2 \times \frac{22}{7} \times 5.25 \times 5.25$$

Thus, the outer surface area of the bowl = 173.25 sq.cm.

Exercise 8.1

1. A solid right circular cylinder has radius of 14 cm and height of 8 cm. Find its curved surface area and total surface area.

2. The total surface area of a solid right circular cylinder is 660 sq.cm. If its diameter of the base is 14 cm, find the height and curved surface area of the cylinder.

3. Curved surface area and circumference at the base of a solid right circular cylinder are 4400 sq.cm and 110 cm respectively. Find its height and diameter.

4. A mansion has 12 right cylindrical pillars each having radius 50 cm and height 3.5 m. Find the cost to paint the lateral surface of the pillars at ₹ 20 per square metre.

5. The total surface area of a solid right circular cylinder is 231 cm^2. Its curved surface area is two thirds of the total surface area. Find the radius and height of the cylinder.

6. The total surface area of a solid right circular cylinder is 1540 cm^2. If the height is four times the radius of the base, then find the height of the cylinder.`

7. The radii of two right circular cylinders are in the ratio of 3 : 2 and their heights are in the ratio 5 : 3. Find the ratio of their curved surface areas.

8. The external surface area of a hollow cylinder is 540π sq.cm. Its internal diameter is 16 cm and height is 15 cm. Find the total surface area.

9. The external diameter of a cylindrical shaped iron pipe is 25 cm and its length is 20 cm. If the thickness of the pipe is 1cm, find the total surface area of the pipe.

10. The radius and height of a right circular solid cone are 7 cm and 24 cm respectively. Find its curved surface area and total surface area.

11. If the vertical angle and the radius of a right circular cone are 60° and 15 cm respectively, then find its height and slant height.

12. If the circumference of the base of a solid right circular cone is 236 cm and its slant height is 12 cm, find its curved surface area.

13. A heap of paddy is in the form of a cone whose diameter is 4.2 m and height is 2.8 m. If the heap is to be covered exactly by a canvas to protect it from rain, then find the area of the canvas needed.

14. The central angle and radius of a sector of a circular disc are 180° and 21 cm respectively. If the edges of the sector are joined together to make a hollow cone, then find the radius of the cone.

15. Radius and slant height of a solid right circular cone are in the ratio 3:5. If the curved surface area is 60π sq.cm, then find its total surface area.

16. If the curved surface area of solid a sphere is 98.56 cm^2, then find the radius of the sphere..

17. If the curved surface area of a solid hemisphere is 2772 sq.cm, then find its total surface area.

18. Radii of two solid hemispheres are in the ratio 3 : 5. Find the ratio of their curved surface areas and the ratio of their total surface areas.

19. Find the curved surface area and total surface area of a hollow hemisphere whose outer and inner radii are 4.2 cm and 2.1 cm respectively.

20. The inner curved surface area of a hemispherical dome of a building needs to be painted. If the circumference of the base is 17.6m, find the cost of painting it at the rate of ₹5 per sq.m.

8.3 Volume

So far we have seen the problems related to the surface area of some solids. Now we shall learn how to calculate volumes of some familiar solids. Volume is literally the 'amount of space filled'. The volume of a solid is a numerical characteristic of the solid.

For example, if a body can be decomposed into finite set of unit cubes (cubes of unit sides), then the volume is equal to the number of these cubes.

The cube in the figure, has a volume

$$= \text{length} \times \text{width} \times \text{height}$$

$$= 1\text{ cm} \times 1\text{ cm} \times 1\text{ cm} = 1\text{ cm}^3.$$

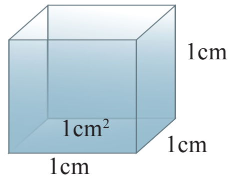


Fig. 8.27

If we say that the volume of an object is 100 cu.cm, then it implies that we need 100 cubes each of 1 cm^3 volume to fill this object completely.

Just like surface area, volume is a positive quantity and is invariant with respect to displacement. Volumes of some solids are illustrated below.

8.3.1 Volume of a right circular cylinder

(i) Volume of a solid right circular cylinder

The volume of a solid right circular cylinder is the product of the base area and height.

That is, the volume of the cylinder, V = Area of the base $\times$ height

$$= \pi r^2 \times h$$

Thus, the volume of a cylinder, $V = \pi r^2 h$ cu. units.

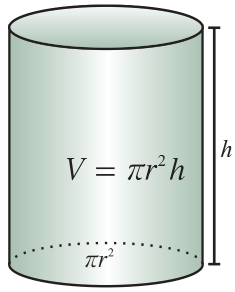


Fig. 8.28

(ii) Volume of a hollow cylinder (Volume of the material used)

Let R and r be the external and internal radii of a hollow right circular cylinder respectively. Let h be its height.

Then, the volume, $V = \left.\begin{matrix}\text{Volume of the}\\ \text{outer cylinder}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of the}\\ \text{inner cylinder}\end{matrix}\right.$

$$= \pi R^2 h - \pi r^2 h$$

Hence, the volume of a hollow cylinder,

$$V = \pi h(R^2 - r^2) \text{ cu. units.}$$

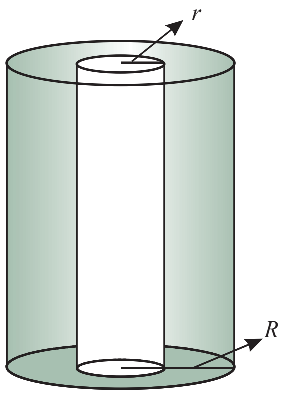


Fig. 8.29

Example 8.12

If the curved surface area of a right circular cylinder is 704 sq.cm, and height is 8 cm, find the volume of the cylinder in litres. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $h = 8$ cm and CSA = 704 sq.cm

Now, CSA = 704

$\Longrightarrow \quad 2\pi rh = 704$

$2 \times \frac{22}{7} \times r \times 8 = 704$

$\therefore \quad r = \frac{704 \times 7}{2 \times 22 \times 8} = 14\text{ cm}$

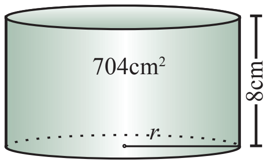


Fig. 8.30

Thus, the volume of the cylinder, $V = \pi r^2 h$

$$= \frac{22}{7} \times 14 \times 14 \times 8$$

$$= 4928 \text{ cu.cm.}$$

Hence, the volume of the cylinder $= 4.928$ litres. (1000 cu.cm = 1 litre)

Example 8.13

A hollow cylindrical iron pipe is of length 28 cm. Its outer and inner diameters are 8 cm and 6 cm respectively. Find the volume of the pipe and weight of the pipe if 1 cu.cm of iron weighs 7 gm.(Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the inner, outer radii and height of the hollow cylindrical pipe respectively.

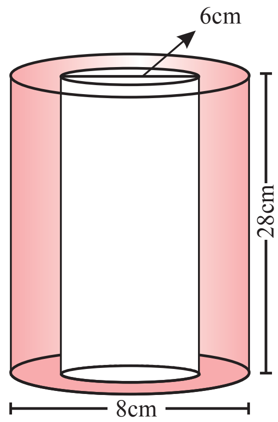


Fig. 8.31

Given that $2r = 6$ cm, $2R = 8$ cm , $h = 28$ cm

Now, the volume of the pipe, $V = \pi \times h \times (R + r)(R - r)$

$$= \frac{22}{7} \times 28 \times (4+3)(4-3)$$

$\therefore$ Volume, $V = 616$ cu. cm

Weight of 1 cu.cm of the metal $= 7$ gm

Weight of the 616 cu. cm of metal $= 7 \times 616$ gm

Thus, the weight of the pipe $= 4.312$ kg.

Example 8.14

Base area and volume of a solid right circular cylinder are 13.86 sq.cm, and 69.3 cu.cm respectively. Find its height and curved surface area.(Take $\pi = \frac{22}{7}$)

Solution Let A and V be the base area and volume of the solid right circular cylinder respectively.

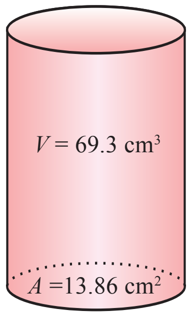


Fig. 8.32

Given that the base area, $A = \pi r^2 = 13.86$ sq.cm and

volume, $V = \pi r^2 h = 69.3$ cu.cm.

Thus, $\pi r^2 h = 69.3$

$\Longrightarrow$ $13.86 \times h = 69.3$

$\therefore$ $h = \frac{69.3}{13.86} = 5$ cm.

Now, the base area $= \pi r^2 = 13.86$

$$\frac{22}{7} \times r^2 = 13.86$$

$$r^2 = 13.86 \times \frac{7}{22} = 4.41 \quad \Longrightarrow \quad r = \sqrt{4.41} = 2.1 \text{ cm.}$$

Now, Curved surface area, CSA $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times 2.1 \times 5$$

Thus, CSA $= 66$ sq.cm.

8.3.2 Volume of a right circular cone

Let r and h be the base radius and the height of a right circular cone respectively.

The volume V of the cone is given by the formula: $V = \frac{1}{3} \times \pi r^2 h$ cu. units. To justify this formula, let us perform the following activity.

Activity

Make a hollow cone and a hollow cylinder like in the figure given below with the same height and same radius.Now, practically we can find out the volume of the cone by doing the process given below. Fill the cone with sand or liquid and then pour it into the cylinder. Continuing this experiment, we see that the cylinder will be filled completely by sand / liquid at the third time.

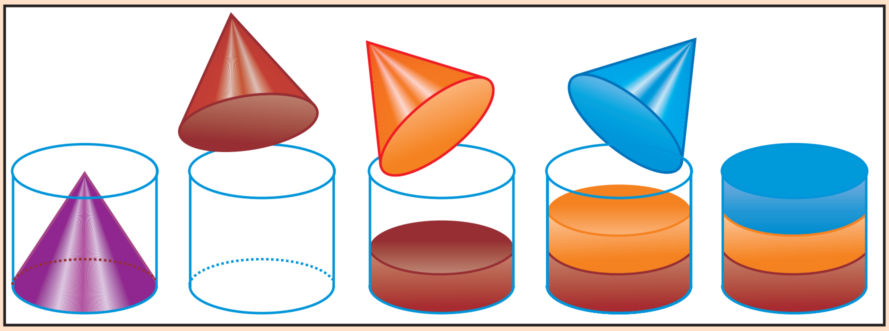


Fig. 8.33

From this simple activity, if r and h are the radius and height of the cylinder, then we find that $3 \times$(Volume of the cone) = Volume of the cylinder $= \pi r^2 h$

Thus, the volume of the cone $= \frac{1}{3} \times \pi r^2 h$ cu. units.

Example 8.15

The volume of a solid right circular cone is 4928 cu. cm. If its height is 24 cm, then find the radius of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r, h and V be the radius, height and volume of a solid cone respectively.

Given that $V = 4928$ cu.cm and $h = 24$ cm

Thus, we have $\frac{1}{3}\pi r^2 h = 4928$

$$\Longrightarrow \frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 4928$$

$$\Longrightarrow r^2 = \frac{4928 \times 3 \times 7}{22 \times 24} = 196.$$

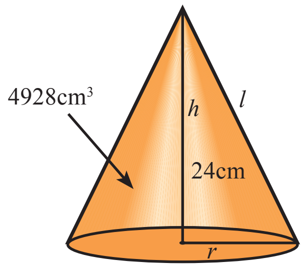


Fig. 8.34

Thus, the base radius of the cone, $r = \sqrt{196} = 14$ cm.

8.3.3 Volume of a Frustum of a Cone

Let us consider a right circular solid cone and cut it into two solids so as to obtain a smaller right circular cone. The other portion of the cone is called frustum of the cone. This is illustrated in the following activity.

Activity

Take some clay and form a right circular cone. Cut it with a knife parallel to its base. Remove the smaller cone. What are you left with? The left out portion of the solid cone is called frustum of the cone. The Latin word frustum means "piece cut off" and its plural is frusta.

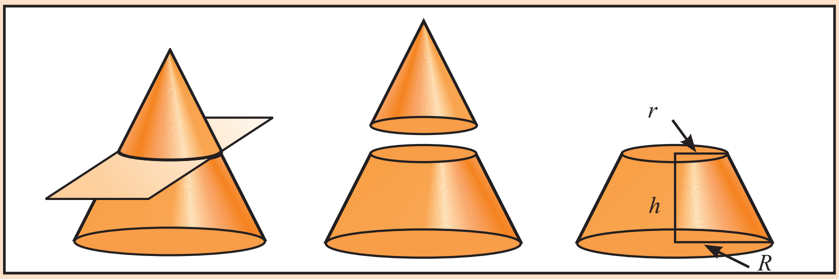


Fig. 8.35

Hence, if a solid right circular cone is sliced with a plane parallel to its base , the part of the cone containing the base is called a frustum of the cone. Thus a frustum has two circular discs, one at the bottom and the other at the top of it.

Let us find the volume of a frustum of a cone.

The volume of a frustum of a cone is nothing but the difference between volumes of two right circular cones. (See Fig. 8.35) Consider a frustum of a solid right circular cone.

Let R be the radius of the given cone. Let r and x be the radius and the height of the smaller cone obtained after removal of the frustum from the given cone.

Let h be the height of the frustum.

$$\text{Now, } \left.\begin{array}{l}\text{the volume of the}\\ \text{frustum of the cone}\end{array}\right\},\ V = \left.\begin{array}{c}\text{Volume of the}\\ \text{given cone}\end{array}\right\} - \left\{\begin{array}{l}\text{Volume of the}\\ \text{smaller cone}\end{array}\right.$$

$$= \frac{1}{3}\times\pi\times R^2\times(x+h) - \frac{1}{3}\times\pi\times r^2\times x$$

$$\text{Thus, } V = \frac{1}{3}\pi[x(R^2-r^2)+R^2h]. \qquad (1)$$

From the Fig. 8.36 we see that $\Delta BFE \sim \Delta DGE$

$$\therefore \quad \frac{BF}{DG} = \frac{FE}{GE}$$

$$\Longrightarrow \quad \frac{R}{r} = \frac{x+h}{x}$$

$$\Longrightarrow \quad Rx - rx = rh$$
$$\Longrightarrow \quad x(R - r) = rh$$

Thus, we get $\quad x = \dfrac{rh}{R - r} \qquad (2)$

Now, (1) $\Longrightarrow \quad V = \dfrac{1}{3}\pi[x(R^2 - r^2) + R^2h]$

$$\Longrightarrow \quad = \frac{1}{3}\pi[x(R - r)(R + r) + R^2h]$$

$$\Longrightarrow \quad = \frac{1}{3}\pi[rh(R + r) + R^2h] \text{ using (2)}$$

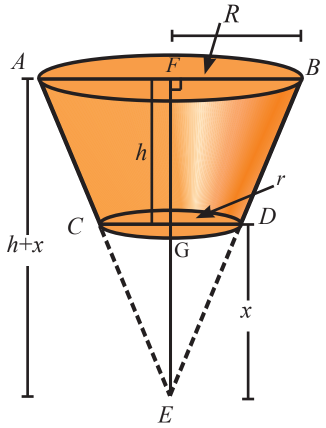


Fig. 8.36

Hence, the volume of the frustum of the cone,

$$V = \frac{1}{3}\pi h(R^2 + r^2 + Rr) \text{ cu. units.}$$

Note

* Curved surface area of a frustum of a cone $= \pi(R + r)l$, where $l = \sqrt{h^2 + (R - r)^2}$

* Total surface area of a frustum of a the cone $= \pi l(R + r) + \pi R^2 + \pi r^2, l = \sqrt{h^2 + (R - r)^2}$

(* Not to be used for examination purpose)

Example 8.16

The radii of two circular ends of a frustum shaped bucket are 15 cm and 8 cm. If its depth is 63 cm, find the capacity of the bucket in litres. (Take $\pi = \dfrac{22}{7}$)

Solution Let R and r are the radii of the circular ends at the top and bottom and h be the depth of the bucket respectively.

Given that $R = 15$ cm , $r = 8$ cm and $h = 63$ cm.

The volume of the bucket (frustum)

$$= \frac{1}{3}\pi h(R^2 + r^2 + Rr)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 63 \times (15^2 + 8^2 + 15 \times 8)$$

$$= 26994 \text{ cu.cm.}$$

$$= \frac{26994}{1000} \text{ litres} \qquad (1000 \text{ cu.cm} = 1 \text{ litre})$$

Fig. 8.37

Thus, the capacity of the bucket = 26.994 litres.

8.3.4 Volume of a Sphere

(i) Volume of a Solid Sphere

The following simple experiment justifies the formula for volume of a sphere,

$$V = \frac{4}{3}\pi r^3 \text{ cu.units.}$$

Activity

Take a cylindrical shaped container of radius R and height H. Fill the container with water. Immerse a solid sphere of radius r, where $R > r$, in the container and fill the displaced water into another cylindrical shaped container of radius r and height H. The height of the water level is equal to $\frac{4}{3}$ times of its radius ($h = \frac{4}{3}r$). Now, the volume of the solid sphere is same as that of the displaced water.

Volume of the displaced water, V $= \text{Base area x Height}$

$$= \pi r^2 \times \frac{4}{3}r \text{ (here, height of the water level } h = \frac{4}{3}r)$$

$$= \frac{4}{3}\pi r^3$$

Thus, the volume of the sphere, $V = \frac{4}{3}\pi r^3$ cu.units.

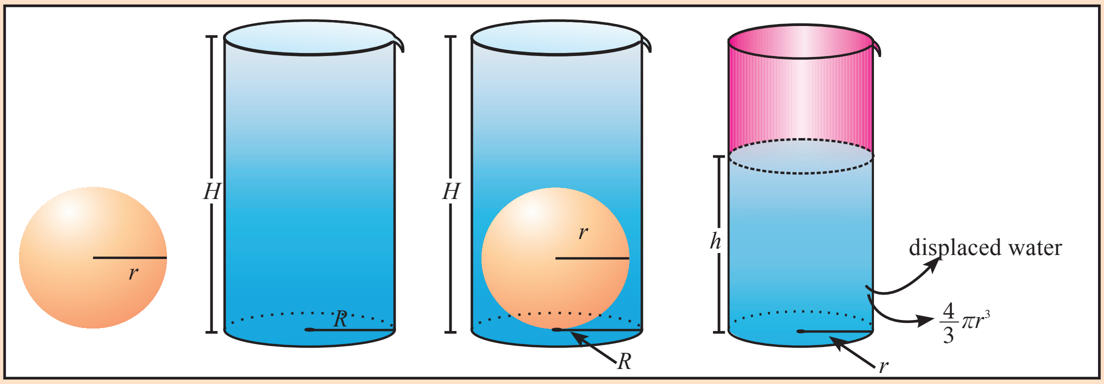


Fig. 8.38

(ii) Volume of a hollow sphere (Volume of the material used)

If the inner and outer radius of a hollow sphere are r and R respectively, then

$$\left.\begin{matrix}\text{Volume of the}\\ \text{hollow sphere}\end{matrix}\right\} = \left.\begin{matrix}\text{Volume of the}\\ \text{outer sphere}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of the}\\ \text{inner sphere}\end{matrix}\right.$$

$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$

$\therefore$ Volume of hollow sphere $= \frac{4}{3}\pi(R^3 - r^3)$ cu. units.

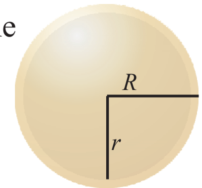


Fig. 8.39

(iii) Volume of a solid hemisphere

Volume of the solid hemisphere $= \frac{1}{2} \times$ volume of the sphere

$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi r^3 \text{ cu.units.}$$

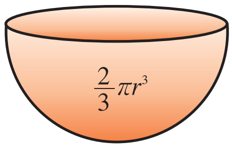


Fig. 8.40

(iv) Volume of a hollow hemisphere (Volume of the material used)

$$\left.\begin{matrix}\text{Volume of a hollow}\\ \text{hemisphere}\end{matrix}\right\} = \left.\begin{matrix}\text{Volume of outer}\\ \text{hemisphere}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of inner}\\ \text{hemisphere}\end{matrix}\right.$$

$$= \frac{2}{3} \times \pi \times R^3 - \frac{2}{3} \times \pi \times r^3$$

$$= \frac{2}{3}\pi(R^3 - r^3) \text{ cu.units.}$$

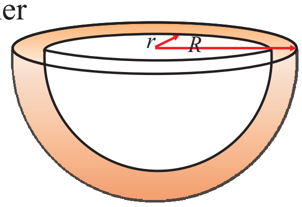


Fig. 8.41

Example 8.17

Find the volume of a sphere-shaped metallic shot-put having diameter of 8.4 cm. (Take $\pi = \frac{22}{7}$)

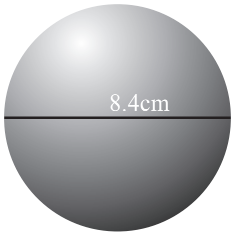


Fig. 8.42

Solution Let r be radius of the metallic shot-put.

Now, $2r = 8.4$ cm $\Longrightarrow$ $r = 4.2$ cm

Volume of the shot-put, $V = \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{42}{10} \times \frac{42}{10} \times \frac{42}{10}$$

Thus, the volume of the shot-put $= 310.464$ cu.cm.

Example 8.18

A cone, a hemisphere and cylinder have equal bases. If the heights of the cone and a cylinder are equal and are same as the common radius, then find the ratio of their respective volumes.

Solution Let r be the common radius of the cone, hemisphere and cylinder.

Let h be the common height of the cone and cylinder.

Given that $r = h$

Let V_1, V_2 and V_3 be the volumes of the cone, hemisphere and cylinder respectively.

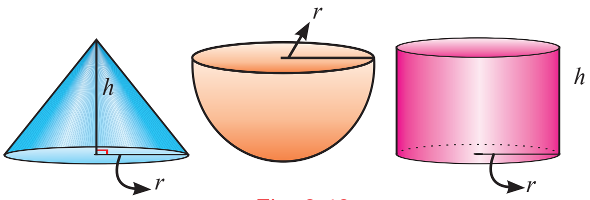


Fig. 8.43

Now, $V_1 : V_2 : V_3 = \frac{1}{3}\pi r^2 h : \frac{2}{3}\pi r^3 : \pi r^2 h$

$\Longrightarrow$ $= \frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3$ (here, $r = h$)

$\Longrightarrow V_1 : V_2 : V_3 = \frac{1}{3} : \frac{2}{3} : 1$

Hence, the required ratio is $1 : 2 : 3$.

Example 8.19

If the volume of a solid sphere is $7241\frac{1}{7}$ cu.cm, then find its radius. (Take $\pi = \frac{22}{7}$)

Solution Let r and V be the radius and volume of the solid sphere respectively.

Given that $V = 7241\frac{1}{7}$ cu.cm

$\Longrightarrow$ $\frac{4}{3}\pi r^3 = \frac{50688}{7}$

$\Longrightarrow$ $\frac{4}{3} \times \frac{22}{7} \times r^3 = \frac{50688}{7}$

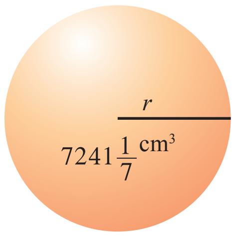


Fig. 8.44

$$r^3 = \frac{50688}{7} \times \frac{3 \times 7}{4 \times 22}$$

$$= 1728 = 4^3 \times 3^3$$

Thus, the radius of the sphere, $r = 12$ cm.

Example 8.20

Volume of a hollow sphere is $\frac{11352}{7}$ cm^3. If the outer radius is 8 cm, find the inner radius of the sphere. (Take $\pi = \frac{22}{7}$)

Solution Let R and r be the outer and inner radii of the hollow sphere respectively.

Let V be the volume of the hollow sphere.

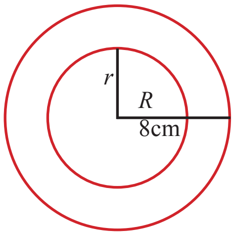


Fig. 8.45

Now, given that $V = \frac{11352}{7}$ cm^3

$$\Longrightarrow \quad \frac{4}{3}\pi(R^3 - r^3) = \frac{11352}{7}$$

$$\Longrightarrow \quad \frac{4}{3} \times \frac{22}{7}(8^3 - r^3) = \frac{11352}{7}$$

$$512 - r^3 = 387 \Longrightarrow r^3 = 125 = 5^3$$

Hence, the inner radius, $r = 5$ cm.

Exercise 8.2

1. Find the volume of a solid cylinder whose radius is 14 cm and height 30 cm.
2. A patient in a hospital is given soup daily in a cylindrical bowl of diameter 7 cm. If the bowl is filled with soup to a height of 4 cm, then find the quantity of soup to be prepared daily in the hospital to serve 250 patients?
3. The sum of the base radius and the height of a solid right circular solid cylinder is 37 cm. If the total surface area of the cylinder is 1628 sq.cm, then find the volume of the cylinder.
4. Volume of a solid cylinder is 62.37 cu.cm. Find the radius if its height is 4.5 cm.
5. The radii of two right circular cylinders are in the ratio 2 : 3. Find the ratio of their volumes if their heights are in the ratio 5 : 3.
6. The radius and height of a cylinder are in the ratio 5 : 7. If its volume is 4400 cu.cm, find the radius of the cylinder.
7. A rectangular sheet of metal foil with dimension 66 cm $\times$ 12 cm is rolled to form a cylinder of height 12 cm. Find the volume of the cylinder.
8. A lead pencil is in the shape of right circular cylinder. The pencil is 28 cm long and its radius is 3 mm. If the lead is of radius 1 mm, then find the volume of the wood used in the pencil.

9. Radius and slant height of a cone are 20 cm and 29 cm respectively. Find its volume.

10. The circumference of the base of a 12 m high wooden solid cone is 44 m. Find the volume.

11. A vessel is in the form of a frustum of a cone. Its radius at one end and the height are 8 cm and 14 cm respectively. If its volume is $\frac{5676}{3}$ cm^3, then find the radius at the other end.

12. The perimeter of the ends of a frustum of a cone are 44 cm and 8.4π cm. If the depth is 14 cm., then find its volume.

13. A right angled $\triangle ABC$ with sides 5 cm, 12 cm and 13 cm is revolved about the fixed side of 12 cm. Find the volume of the solid generated.

14. The radius and height of a right circular cone are in the ratio 2 : 3. Find the slant height if its volume is 100.48 cu.cm. (Take $\pi = 3.14$)

15. The volume of a cone with circular base is 216π cu.cm. If the base radius is 9 cm, then find the height of the cone.

16. Find the mass of 200 steel spherical ball bearings, each of which has radius 0.7 cm, given that the density of steel is 7.95 g/cm^3. (Mass = Volume $\times$ Density)

17. The outer and the inner radii of a hollow sphere are 12 cm and 10 cm. Find its volume.

18. The volume of a solid hemisphere is 1152π cu.cm. Find its curved surface area.

19. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 14 cm.

20. The radius of a spherical balloon increases from 7 cm to 14 cm as air is being pumped into it. Find the ratio of volumes of the balloon in the two cases.

8.4 Combination of Solids

In our daily life we observe many objects like toys, vehicles, vessels, tools, etc., which are combination of two or more solids.

How can we find the surface areas and volumes of combination of solids?

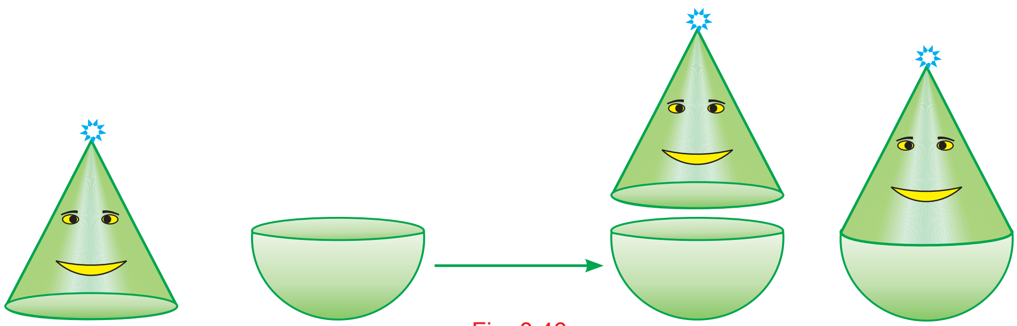


Fig. 8.46

The total surface area of the combination of solids need not be the sum of the surface areas of the solids which are combined together. However, in the above figure, the total surface area of the combined solid is equal to the sum of the curved surface area of the hemisphere and curved surface area of the cone. But the volume of the combined solid is equal to the sum of the volumes of the solids which are combined together. Thus, from the figure we have,

$$\text{The total surface area of the solid} = \left.\begin{matrix}\text{Curved surface area}\\ \text{of the hemisphere}\end{matrix}\right\} + \left\{\begin{matrix}\text{Curved surface area}\\ \text{of the cone}\end{matrix}\right.$$

The total volume of the solid = Volume of the hemisphere + Volume of the cone.

Example 8.21

A solid wooden toy is in the form of a cone surmounted on a hemisphere. If the radii of the hemisphere and the base of the cone are 3.5 cm each and the total height of the toy is 17.5 cm, then find the volume of wood used in the toy. (Take $\pi = \frac{22}{7}$)

Solution Hemispherical portion :

Radius, $r = 3.5$ cm

Conical portion :

Radius, $r = 3.5$ cm

Height, $h = 17.5 - 3.5 = 14$ cm

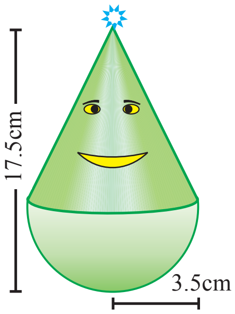


Fig. 8.47

Volume of the wood = Volume of the hemisphere + Volume of the cone

$$= \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi r^2}{3}(2r + h)$$

$$= \frac{22}{7} \times \frac{3.5 \times 3.5}{3} \times (2 \times 3.5 + 14) = 269.5$$

Hence, the volume of the wood used in the toy = 269.5 cu.cm.

Example 8.22

A cup is in the form of a hemisphere surmounted by a cylinder. The height of the cylindrical portion is 8 cm and the total height of the cup is 11.5 cm. Find the total surface area of the cup. (Take $\pi = \frac{22}{7}$)

Solution Hemispherical portion

Radius, r = Total height – 8

$\Longrightarrow r = 11.5 - 8 = 3.5$ cm

Cylindrical portion

Height, $h = 8$ cm.

Thus, radius $r = 3.5 \text{ cm} = \frac{7}{2}$ cm

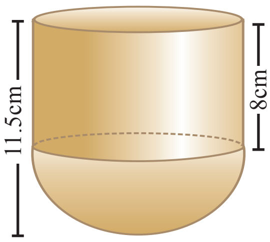


Fig. 8.48

$$\text{Total surface area of the cup} = \left\{\begin{matrix}\text{CSA of the hemispherical portion}\\ +\ \text{CSA of the cylindrical portion}\end{matrix}\right.$$

$$= 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$$

$$= 2 \times \frac{22}{7} \times \frac{7}{2}\left(\frac{7}{2} + 8\right)$$

$\therefore$ Total surface area of the cup = 253 sq.cm.

Example 8.23

A circus tent is to be erected in the form of a cone surmounted on a cylinder. The total height of the tent is 49 m. Diameter of the base is 42 m and height of the cylinder is 21 m. Find the cost of canvas needed to make the tent, if the cost of canvas is ₹12.50/m^2. (Take $\pi = \frac{22}{7}$)

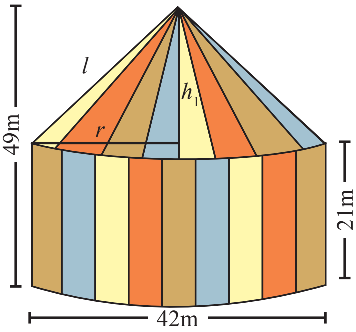


Fig. 8.49

Solution

Cylindrical Part	Conical Part
Diameter, $2r = 42$ m	Radius, $r = 21$ m
Radius, $r = 21$ m	Height, $h_1 = 49 - 21 = 28$ m
Height, $h = 21$ m	Slant height, $l = \sqrt{h_1^2 + r^2}$
	$= \sqrt{28^2 + 21^2}$
	$= 7\sqrt{4^2 + 3^2} = 35$ m

Total area of the canvas needed = CSA of the cylindrical part + CSA of the conical part

$$= 2\pi rh + \pi rl = \pi r(2h + l)$$

$$= \frac{22}{7} \times 21(2 \times 21 + 35) = 5082$$

Therefore, area of the canvas = 5082 m^2

Now, the cost of the canvas per sq.m = ₹12.50

Thus, the total cost of the canvas = 5082×12.5 = ₹63525.

Example 8.24

A hollow sphere of external and internal diameters of 8 cm and 4 cm respectively is melted and made into another solid in the shape of a right circular cone of base diameter of 8 cm. Find the height of the cone.

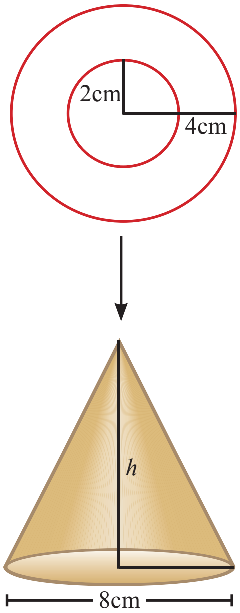


Fig. 8.50

Solution Let R and r be the external and internal radii of the hollow sphere.

Let h and r_1 be the height and the radius of the cone to be made.

Hollow Sphere: External	Hollow Sphere: Internal	Cone
$2R = 8$ cm	$2r = 4$ cm	$2r_1 = 8$
$\Longrightarrow R = 4$ cm	$\Longrightarrow r = 2$ cm	$\Longrightarrow r_1 = 4$

When the hollow sphere is melted and made into a solid cone, we have

Volume of the cone = Volume of the hollow sphere

$$\Longrightarrow \quad \frac{1}{3}\pi r_1^2 h = \frac{4}{3}\pi[R^3 - r^3]$$

$$\Longrightarrow \qquad \frac{1}{3}\times\pi\times4^2\times h = \frac{4}{3}\times\pi\times(4^3-2^3)$$

$$\Longrightarrow \qquad h = \frac{64-8}{4} = 14$$

Hence, the height of the cone $h = 14$ cm.

Example 8.25

Spherical shaped marbles of diameter 1.4 cm each, are dropped into a cylindrical beaker of diameter 7 cm containing some water. Find the number of marbles that should be dropped into the beaker so that the water level rises by 5.6 cm.

Solution Let n be the number of marbles needed. Let r_1 and r_2 be the radii of the marbles and cylindrical beaker respectively.

Marbles	Cylindrical Beaker
Diameter, $2r_1 = 1.4$ cm	Diameter, $2r_2 = 7$ cm
Radius $r_1 = 0.7$ cm	Radius, $r_2 = \frac{7}{2}$ cm

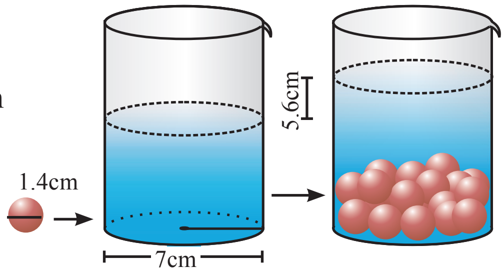


Fig. 8.51

Let h be the height of the water level raised.

Then, $h = 5.6$ cm

After the marbles are dropped into the beaker,

Volume of water raised = Volume of n marbles

$$\Longrightarrow \quad \pi r_2^2 h = n\times\frac{4}{3}\pi r_1^3$$

Thus, $$n = \frac{3r_2^2 h}{4r_1^3}$$

$$n = \frac{3\times\frac{7}{2}\times\frac{7}{2}\times5.6}{4\times\frac{7}{10}\times\frac{7}{10}\times\frac{7}{10}} = 150.$$

$\therefore$ The number of marbles needed is 150.

Example 8.26

Water is flowing at the rate of 15 km / hr through a cylindrical pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. In how many hours will the water level in the tank raise by 21 cm? (Take $\pi = \frac{22}{7}$)

Solution Speed of water = 15 km / hr

= 15000 m / hr

Diameter of the pipe, $2r = 14$ cm

Thus, $r = \frac{7}{100}$ m.

Let h be the water level to be raised.

Thus, $h = 21\text{ cm} = \frac{21}{100}$ m

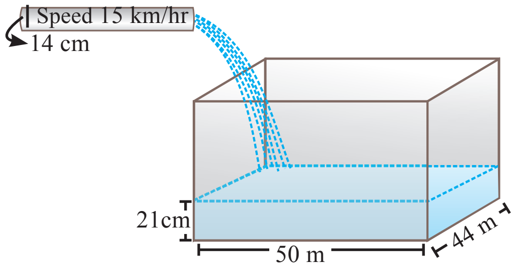


Fig. 8.52

Now, the volume of water discharged

$$= \text{Cross section area of the pipe} \times \text{Time} \times \text{Speed}$$

Volume of water discharged in one hour

$$= \pi r^2 \times 1 \times 15000$$
$$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 15000 \text{ cu.m}$$

Volume of required quantity of water in the tank is,

$$lbh = 50 \times 44 \times \frac{21}{100}$$

Assume that T hours are needed to get the required quantity of water.

$\therefore$ $\left.\begin{array}{r}\text{Volume of water discharged}\\ \text{in T hours}\end{array}\right\} = \text{Required quantity of water in the tank}$

$$\Longrightarrow \frac{22}{7} \times \left(\frac{7}{100}\right)^2 \times T \times 15000 = 50 \times 44 \times \frac{21}{100}$$

Thus, $T = 2$ hours.

Hence, it will take 2 hours to raise the required water level.

Example 8.27

A cuboid shaped slab of iron whose dimensions are 55 cm$\times$40 cm$\times$15 cm is melted and recast into a pipe. The outer diameter and thickness of the pipe are 8 cm and 1 cm respectively. Find the length of the pipe. (Take $\pi = \frac{22}{7}$)

Solution Let h_1 be the length of the pipe.

Let R and r be the outer and inner radii of the pipe respectively.

Iron slab: Let $lbh = 55 \times 40 \times 15$.

Iron pipe:

Outer diameter, $2R = 8$ cm

$\therefore$ Outer radius, $R = 4$ cm

Thickness, $w = 1$ cm

$\therefore$ Inner radius, $r = R - w = 4 - 1 = 3$ cm

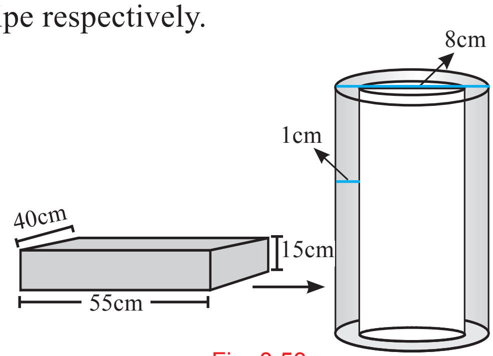


Fig. 8.53

Now, the volume of the iron pipe = Volume of iron slab

$$\Longrightarrow \pi h_1 (R + r)(R - r) = lbh$$

That is, $\frac{22}{7} \times h_1 (4 + 3)(4 - 3) = 55 \times 40 \times 15$

Thus, the length of the pipe, $h_1 = 1500$ cm $= 15$ m.

Exercise 8.3

1. A play-top is in the form of a hemisphere surmounted on a cone. The diameter of the hemisphere is 3.6 cm. The total height of the play-top is 4.2 cm. Find its total surface area.

2. A solid is in the shape of a cylinder surmounted on a hemisphere. If the diameter and the total height of the solid are 21 cm, 25.5 cm respectively, then find its volume.

3. A capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends. If the length of the entire capsule is 14 mm and the diameter of the capsule is 5 mm, find its surface area.

4. A tent is in the shape of a right circular cylinder surmounted by a cone. The total height and the diameter of the base are 13.5 m and 28 m. If the height of the cylindrical portion is 3 m, find the total surface area of the tent.

5. Using clay, a student made a right circular cone of height 48 cm and base radius 12 cm. Another student reshapes it in the form of a sphere. Find the radius of the sphere.

6. The radius of a solid sphere is 24 cm. It is melted and drawn into a long wire of uniform cross section. Find the length of the wire if its radius is 1.2 mm.

7. A right circular conical vessel whose internal radius is 5 cm and height is 24 cm is full of water. The water is emptied into an empty cylindrical vessel with internal radius 10 cm. Find the height of the water level in the cylindrical vessel.

8. A solid sphere of diameter 6 cm is dropped into a right circular cylindrical vessel with diameter 12 cm, which is partly filled with water. If the sphere is completely submerged in water, how much does the water level in the cylindrical vessel increase?.

9. Through a cylindrical pipe of internal radius 7 cm, water flows out at the rate of 5 cm/sec. Calculate the volume of water (in litres) discharged through the pipe in half an hour.

10. Water in a cylindrical tank of diameter 4 m and height 10 m is released through a cylindrical pipe of diameter 10 cm at the rate of 2.5 Km/hr. How much time will it take to empty the half of the tank? Assume that the tank is full of water to begin with.

11. A spherical solid material of radius 18 cm is melted and recast into three small solid spherical spheres of different sizes. If the radii of two spheres are 2cm and 12 cm, find the radius of the third sphere.

12. A hollow cylindrical pipe is of length 40 cm. Its internal and external radii are 4 cm and 12 cm respectively. It is melted and cast into a solid cylinder of length 20 cm. Find the radius of the new solid.

13. An iron right circular cone of diameter 8 cm and height 12 cm is melted and recast into spherical lead shots each of radius 4 mm. How many lead shots can be made?.

14. A right circular cylinder having diameter 12 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 12 cm and diameter 6 cm, having a hemispherical shape on top. Find the number of such cones which can be filled with the ice cream available.

15. A container with a rectangular base of length 4.4 m and breadth 2 m is used to collect rain water. The height of the water level in the container is 4 cm and the water is transferred into a cylindrical vessel with radius 40 cm. What will be the height of the water level in the cylinder?

16. A cylindrical bucket of height 32 cm and radius 18 cm is filled with sand. The bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap.

17. A cylindrical shaped well of depth 20 m and diameter 14 m is dug. The dug out soil is evenly spread to form a cuboid-platform with base dimension 20 m $\times$ 14 m. Find the height of the platform.

Exercise 8.4

Choose the correct answer

1. The curved surface area of a right circular cylinder of radius 1 cm and height 1 cm is equal to

   (A) π cm^2 (B) 2π cm^2 (C) 3π cm^3 (D) 2 cm^2

2. The total surface area of a solid right circular cylinder whose radius is half of its height h is equal to

   (A) $\frac{3}{2}\pi h$ sq. units (B) $\frac{2}{3}\pi h^2$ sq. units (C) $\frac{3}{2}\pi h^2$ sq.units (D) $\frac{2}{3}\pi h$ sq.units

3. Base area of a right circular cylinder is 80 cm^2. If its height is 5 cm, then the volume is equal to

   (A) 400 cm^3 (B) 16 cm^3 (C) 200 cm^3 (D) $\frac{400}{3}$ cm^3

4. If the total surface area a solid right circular cylinder is 200π cm^2 and its radius is 5 cm, then the sum of its height and radius is

   (A) 20 cm (B) 25 cm (C) 30 cm (D) 15 cm

5. The curved surface area of a right circular cylinder whose radius is a units and height is b units, is equal to

   (A) $\pi a^2 b$ sq.cm (B) $2\pi ab$ sq.cm (C) 2π sq.cm (D) 2 sq.cm

6. Radius and height of a right circular cone and that of a right circular cylinder are respectively, equal. If the volume of the cylinder is 120 cm^3, then the volume of the cone is equal to

   (A) 1200 cm^3 B) 360 cm^3 (C) 40 cm^3 (D) 90 cm^3

7. If the diameter and height of a right circular cone are 12 cm and 8 cm respectively, then the slant height is

(A) 10 cm (B) 20 cm (C) 30 cm (D) 96 cm

8. If the circumference at the base of a right circular cone and the slant height are 120π cm and 10 cm respectively, then the curved surface area of the cone is equal to

(A) 1200π cm^2 (B) 600π cm^2 (C) 300π cm^2 (D) 600 cm^2

9. If the volume and the base area of a right circular cone are 48π cm^3 and 12π cm^2 respectively, then the height of the cone is equal to

(A) 6 cm (B) 8 cm (C) 10 cm (D) 12 cm

10. If the height and the base area of a right circular cone are 5 cm and 48 sq.cm respectively, then the volume of the cone is equal to

(A) 240 cm^3 (B) 120 cm^3 (C) 80 cm^3 (D) 480 cm^3

11. The ratios of the respective heights and the respective radii of two cylinders are 1:2 and 2:1 respectively. Then their respective volumes are in the ratio

(A) 4 : 1 (B) 1 : 4 (C) 2 : 1 (D) 1 : 2

12. If the radius of a sphere is 2 cm , then the curved surface area of the sphere is equal to

(A) 8π cm^2 (B) 16 cm^2 (C) 12π cm^2 (D) 16π cm^2 .

13. The total surface area of a solid hemisphere of diameter 2 cm is equal to

(A) 12 cm^2 (B) 12π cm^2 (C) 4π cm^2 (D) 3π cm^2.

14. If the volume of a sphere is $\frac{9}{16}\pi$ cu.cm, then its radius is

(A) $\frac{4}{3}$ cm (B) $\frac{3}{4}$ cm (C) $\frac{3}{2}$ cm (D) $\frac{2}{3}$ cm.

15. The surface areas of two spheres are in the ratio of 9 : 25. Then their volumes are in the ratio

(A) 81 : 625 (B) 729 : 15625 (C) 27 : 75 (D) 27 : 125.

16. The total surface area of a solid hemisphere whose radius is a units, is equal to

(A) $2\pi a^2$ sq.units (B) $3\pi a^2$ sq.units (C) $3\pi a$ sq.units (D) $3a^2$ sq.units.

17. If the surface area of a sphere is 100π cm^2, then its radius is equal to

(A) 25 cm (B) 100 cm (C) 5 cm (D) 10 cm .

18. If the surface area of a sphere is 36π cm^2, then the volume of the sphere is equal to

(A) 12π cm^3 (B) 36π cm^3 (C) 72π cm^3 (D) 108π cm^3.

19. If the total surface area of a solid hemisphere is 12π cm^2 then its curved surface area is equal to

(A) 6π cm^2 (B) 24π cm^2 (C) 36π cm^2 (D) 8π cm^2.

20. If the radius of a sphere is half of the radius of another sphere, then their respective volumes are in the ratio

(A) 1 : 8 (B) 2: 1 (C) 1 : 2 (D) 8 : 1

21. Curved surface area of solid sphere is 24 cm^2. If the sphere is divided into two hemispheres, then the total surface area of one of the hemispheres is

(A) 12 cm^2 (B) 8 cm^2 (C) 16 cm^2 (D) 18 cm^2

22. Two right circular cones have equal radii. If their slant heights are in the ratio 4 : 3, then their respective curved surface areas are in the ratio

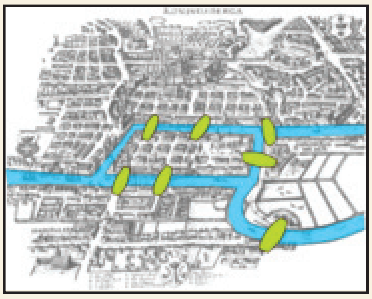
(A) 16 : 9 (B) 2 : 3 (C) 4 : 3 (D) 3 : 4

Do you know?

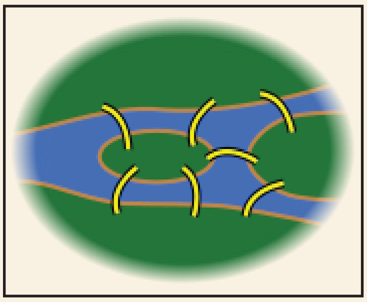
The Seven Bridges of Königsberg is a notable historical problem in mathematics. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.(See Figure)

The problem was to find a route through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time (one could not walk half way onto the bridge and then turn around and later cross the other half from the other side).

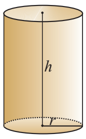
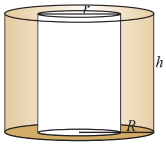
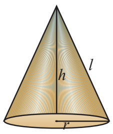
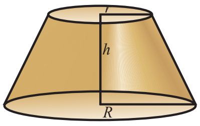
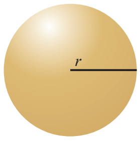
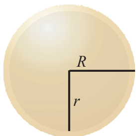
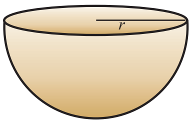
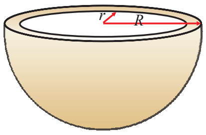
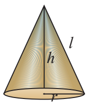
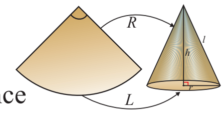
Leonhard Euler in 1735 proved that the problem has no solution. Its negative resolution by Euler laid the foundations of graph theory and presaged the idea of topology.



$\Longleftrightarrow$



Points to Remember

Sl. No	Name	Figure	Lateral or Curved Surface Area (sq.units)	Total Surface Area (sq.units)	Volume (cu.units)
1	**Solid right circular cylinder**		$2\pi rh$	$2\pi r(h+r)$	$\pi r^2 h$
2	**Right circular hollow cylinder**		$2\pi h(R+r)$	$2\pi(R+r)(R-r+h)$	Volume of the material used $\pi R^2 h - \pi r^2 h$ $= \pi h(R^2 - r^2)$ $= \pi h(R+r)(R-r)$
3	**Solid right circular cone**		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2 h$
4	**Frustum**		- - - - - - -	- - - - - - - - - - - - - - -	$\frac{1}{3}\pi h(R^2 + r^2 + Rr)$
5	**Sphere**		$4\pi r^2$	- - -	$\frac{4}{3}\pi r^3$
6	**Hollow sphere**		- - -	- - -	Volume of the material used $\frac{4}{3}\pi(R^3 - r^3)$
7	**Solid Hemisphere**		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$
8	**Hollow Hemisphere**		$2\pi(R^2 + r^2)$	$2\pi(R^2+r^2)+\pi(R^2-r^2)$ $= \pi(3R^2 + r^2)$	Volume of the material used $\frac{2}{3}\pi(R^3 - r^3)$
9	**A sector of a circle converted into a Cone** CSA of a cone = Area of the sector $\pi rl = \frac{\theta}{360} \times \pi r^2$ Length of the sector = Base circumference of the cone	 	$l = \sqrt{h^2 + r^2}$ $h = \sqrt{l^2 - r^2}$ $r = \sqrt{l^2 - h^2}$	10. Volume of water flows out through a pipe = {Cross section area $\times$ Speed $\times$ Time } 11. No. of new solids obtained by recasting $= \dfrac{\text{Volume of the solid which is melted}}{\text{volume of one solid which is made}}$	
12	Conversions	1 m^3 = 1000 litres , 1 d.m^3 = 1 litre , 1000 cm^3 = 1 litre , 1000 litres = 1 *kl*			

9

PRACTICAL GEOMETRY

Give me a place to stand, and I shall move the earth

-Archimedes

- Introduction
- Tangents
- Triangles
- Cyclic Quadrilaterals

Brahmagupta

(598-668 AD)

India

(Great Scientist of Ancient India)

Brahmagupta wrote the book "Brahmasphuta Siddhanta". His most famous result in geometry is a formula for cyclic quadrilateral :

Given the lengths p, q, r and s of the sides of any cyclic quadrilateral, he gave an approximate and an exact formula for the area.

Approximate area is

$$\left(\frac{p+r}{2}\right)\left(\frac{q+s}{2}\right).$$

Exact area is

$$\sqrt{(t-p)(t-q)(t-r)(t-s)},$$

where $2t = p+q+r+s$.

9.1 Introduction

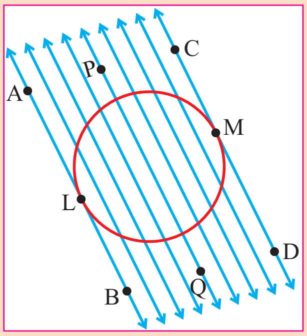
Geometry originated in Egypt as early as 3000 B.C., was used for the measurement of land. Early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes which were developed to meet some practical needs in surveying, construction, astronomy and various other crafts.

Recently there have been several new efforts to reform curricula to make geometry less worthy than its counterparts such as algebra, analysis, etc. But many mathematicians strongly disagree with this reform. In fact, geometry helps in understanding many mathematical ideas in other parts of mathematics. In this chapter, we shall learn how to draw tangents to circles, triangles and cyclic quadrilaterals with the help of given actual measurements.

In class IX, we have studied about various terms related to circle such as chord, segment, sector, etc. Let us recall some of the terms like secant, tangent to a circle through the following activities.

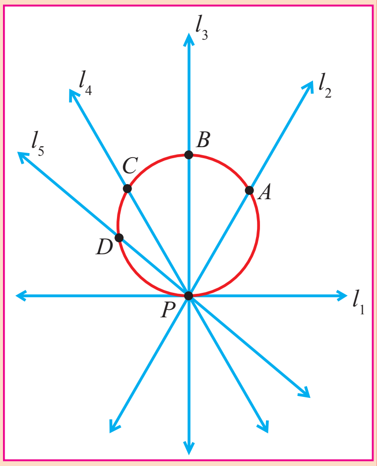
Activity

Take a paper and draw a circle of any radius. Draw a secant PQ to the circle. Now draw as many secants as possible parallel to PQ on both sides of PQ. Note that the points of contact of the secants are coming closer and closer on



either side. You can also note that at one stage, the two points will coincide on both sides. Among the secants parallel to PQ, the straight lines AB and CD, just touch the circle exactly at one point on the circle, say at L and M respectively. These lines AB, CD are called **tangents** to the circle at L, M respectively. We observe that AB is parallel to CD.

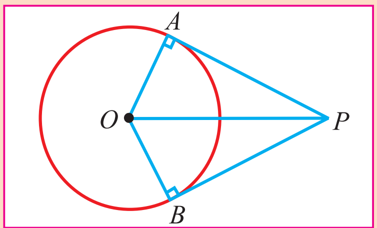
Activity



Let us draw a circle and take a point P on the circle. Draw many lines through the point P as shown in the figure. The straight lines which are passing through P, have two contact points on the circle. The straight lines l_2, l_3, l_4 and l_5 meet the circle at A, B, C and D respectively. So these lines l_2, l_3, l_4, l_5 are the secants to the circle. But the line l_1 touches the circle exactly at one point P. Now the line l_1 is called the **tangent** to the circle at P.

We know that in a circle, the radius drawn at the point of contact is perpendicular to the tangent at that point.

Let AP be a tangent at A drawn from an external point P to a circle



In a right angled ΔOPA , $OA \perp AP$

$$OP^2 = OA^2 + AP^2 \quad \text{[By Pythagoras theorem]}$$

$$AP = \sqrt{OP^2 - OA^2}.$$

9.2 Construction of tangents to a circle

Now let us learn how to draw a tangent to a circle

(i) using centre

(ii) using tangent-chord theorem .

9.2.1 Construction of a tangent to a circle (using the centre)

Result

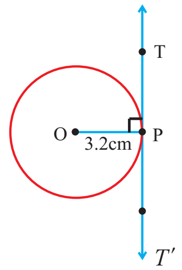
In a circle, the radius drawn at the point of contact is perpendicular to the tangent at that point.

Example 9.1

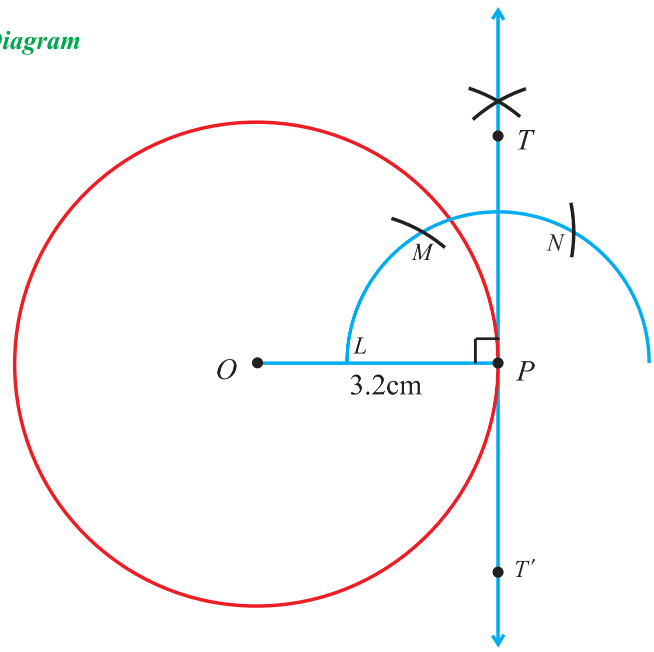
Draw a circle of radius 3.2cm. Take a point P on this circle and draw a tangent at P. (using the centre)

Given: Radius of the circle = 3.2 cm.

Rough Diagram



Fair Diagram



Construction

(i) With O as the centre draw a circle of radius 3.2 cm.

(ii) Take a point P on the circle and join OP.

(iii) Draw an arc of a circle with centre at P cutting OP at L.

(iv) Mark M and N on the arc such that $\overset{\frown}{LM} = \overset{\frown}{MN} = LP$.

(v) Draw the bisector PT of the angle $\angle MPN$.

(vi) Produce TP to T' to get the required tangent $T'PT$.

Remarks

One can draw the perpendicular line PT to the straight line OP through the point P on the circle. Now, PT is the tangent to the circle at the point P.

9.2.2 Construction of a tangent to a circle using the tangent-chord theorem

Result **The tangent-chord theorem**

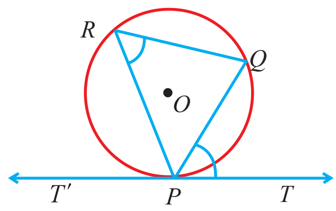
The angle between a chord of a circle and the tangent at one end of the chord is equal to the angle subtended by the chord on the alternate segment of the circle.

Example 9.2

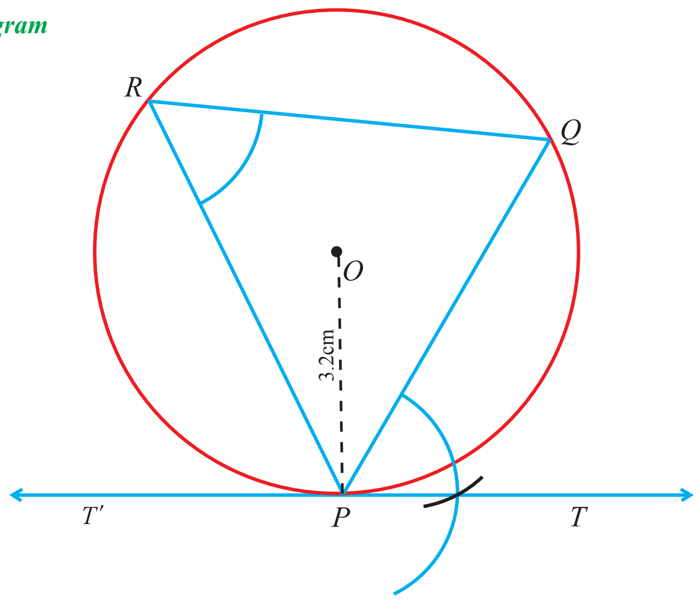
Draw a circle of radius 3.2 cm At a point P on it, draw a tangent to the circle using the tangent-chord theorem.

Given : The radius of the circle = 3.2 cm.

Rough Diagram



Fair Diagram



Construction

(i) With O as the centre, draw a circle of radius 3.2 cm.

(ii) Take a point P on the circle.

(iii) Through P, draw any chord PQ.

(iv) Mark a point R distinct from P and Q on the circle so that P, Q and R are in counter clockwise direction.

(v) Join PR and QR.

(vi) At P, construct $\angle QPT = \angle PRQ$.

(vii) Produce TP to T' to get the required tangent line $T'PT$.

9.2.3 Construction of pair of tangents to a circle from an external point

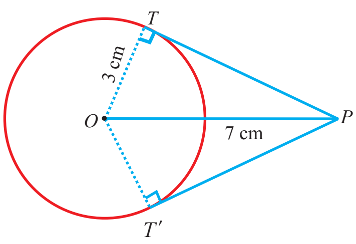
Results
(i) Two tangents can be drawn to a circle from an external point.

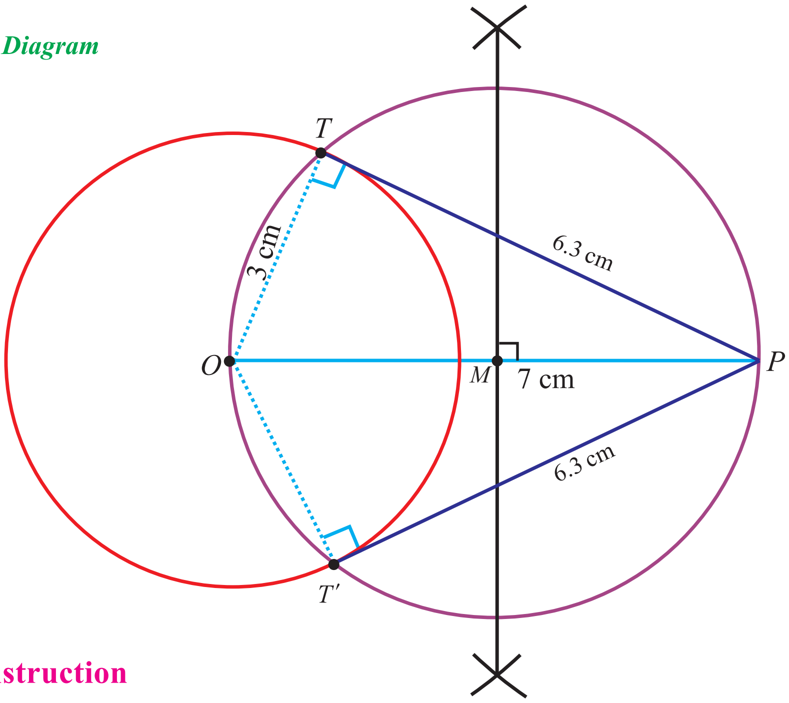
(ii) Diameters subtend 90° on the circumference of a circle.

Example 9.3

Draw a circle of radius 3 cm. From an external point 7 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

Given: Radius of the circle = 3 cm. OP = 7 cm.





Construction

(i) With O as the centre draw a circle of radius 3 cm.

(ii) Mark a point P at a distance of 7 cm from O and join OP.

(iii) Draw the perpendicular bisector of OP. Let it meet OP at M.

(iv) With M as centre and MO as radius, draw another circle.

(v) Let the two circles intersect at T and T'.

(vi) Join PT and PT'. They are the required tangents.

Length of the tangent, PT = 6.3 cm

Verification

In the right angled ΔOPT,

$$PT = \sqrt{OP^2 - OT^2} = \sqrt{7^2 - 3^2}$$

$$= \sqrt{49 - 9} = \sqrt{40} \qquad \therefore \quad PT = 6.3 \text{ cm (approximately).}$$

Exercise 9.1

1. Draw a circle of radius 4.2 cm, and take any point on the circle. Draw the tangent at that point using the centre.

2. Draw a circle of radius 4.8 cm. Take a point on the circle. Draw the tangent at that point using the tangent-chord theorem.

3. Draw a circle of diameter 10 cm. From a point P, 13 cm away from its centre, draw the two tangents PA and PB to the circle, and measure their lengths.

4. Draw the two tangents from a point which is 10 cm away from the centre of a circle of radius 6 cm. Also, measure the lengths of the tangents.

5. Take a point which is 9 cm away from the centre of a circle of radius 3 cm, and draw the two tangents to the circle from that point.

9.3 Construction of triangles

We have already learnt how to construct triangles when sides and angles are given. In this section, let us construct a triangle when

(i) the base, vertical angle and the altitude from the vertex to the base are given.

(ii) the base, vertical angle and the median from the vertex to the base are given.

First, let us describe the way of constructing a segment of a circle on a given line segment containing a given angle.

Construction of a segment of a circle on a given line segment containing an angle θ

Construction

(i) Draw a line segment $\overline{BC}$.

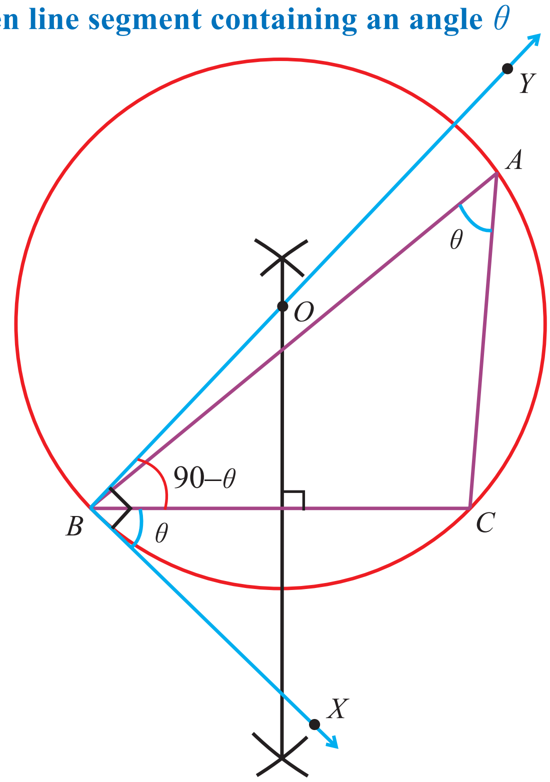
(ii) At B, make $\angle CBX = \theta$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC which meets BY at O.

(v) With O as centre and OB as radius draw a circle.

(vi) Take any point A on the circle. By the tangent-chord theorem, the major arc BAC is the required segment of the circle containing the angle θ.



Construction of a triangle when its base and the vertical angle are given.

We shall describe the various steps involved in the construction of a triangle when its base and the vertical angle are given.

Construction

(i) Draw a line segment AB.

(ii) At A, make the given angle $\angle BAX = \theta$

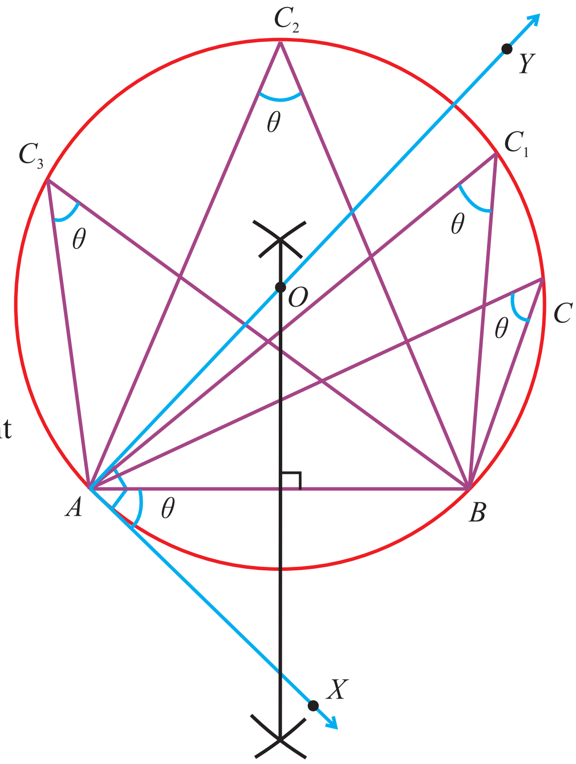
(iii) Draw $AY \perp AX$.

(iv) Draw the perpendicular bisector of AB which meets AY at O.

(v) With O as centre OA as radius, draw a circle.

(vi) Take any point C on the alternate segment of the circle and join AC and BC.

(vii) $\triangle ABC$ is the required triangle.



Now, one can justify that $\triangle ABC$ is one of the triangles, with the given base and the vertical angle.

Note that $AX \perp AY$. Thus, $\angle XAY = 90^\circ$.

Also, $OB = OA$. (the radii of the circle).

AX is the tangent to the circle at A and C is any point on the circle.

Hence, $\angle BAX = \angle ACB$. (tangent-chord theorem).

Remarks

If we take $C_1, C_2, C_3, \ldots$ are points on the circle, then all the triangle $\Delta ABC_1,\ \Delta ABC_2,\ \Delta ABC_3, \cdots$ are with same base and the same vertical angle.

9.3.1 Construction of a triangle when its base, the vertical angle and the altitude from the vertex to the base are given.

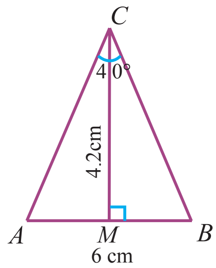
Example 9.4

Construct a ΔABC such that $AB = 6$ cm, $\angle C = 40^\circ$ and the altitude from C to AB is of length 4.2 cm.

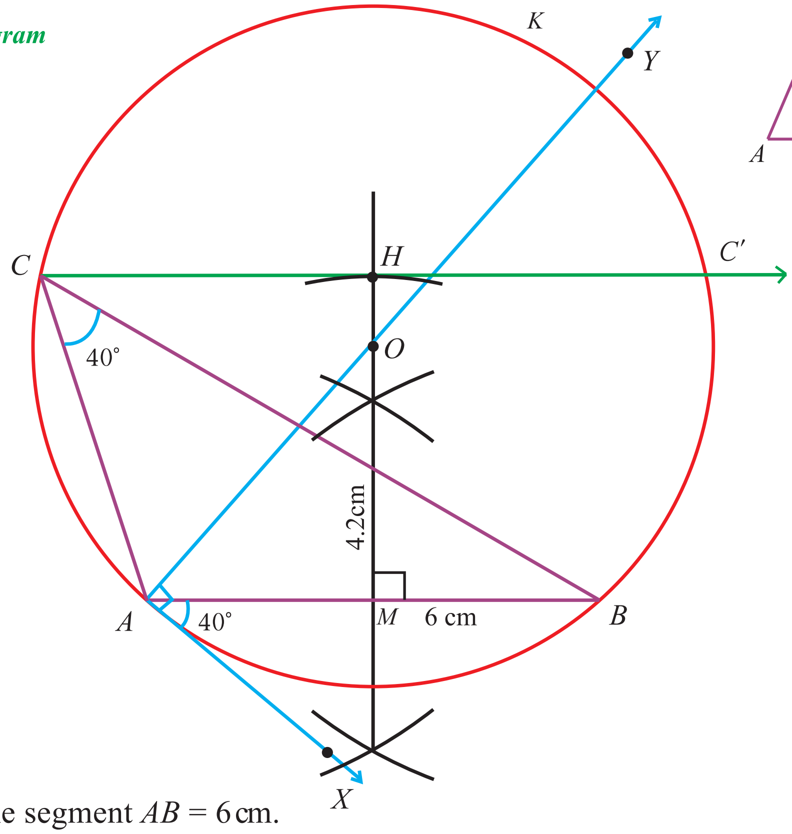
Given : In ΔABC, $AB = 6$ cm, $\angle C = 40^\circ$

The length of the altitude from C to AB is 4.2 cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a line segment $AB = 6$ cm.

(ii) Draw AX such that $\angle BAX = 40^\circ$.

(iii) Draw $AY \perp AX$.

(iv) Draw the perpendicular bisector of AB intersecting AY at O and AB at M.

(v) With O as centre and OA as radius, draw the circle .

(vi) The segment AKB contains the vertical angle 40°.

(vii) On the perpendicular bisector MO, mark a point H such that $MH = 4.2$ cm.

(viii) Draw CHC' parallel to AB meeting the circle at C and at C'.

(ix) Complete the ΔABC, which is one of the required triangles.

Remarks

$\triangle ABC'$ is also another required triangle.

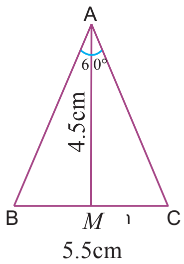
9.3.2 Construction of a triangle when its base, the vertical angle and the median from the vertex to the base are given.

Example 9.5

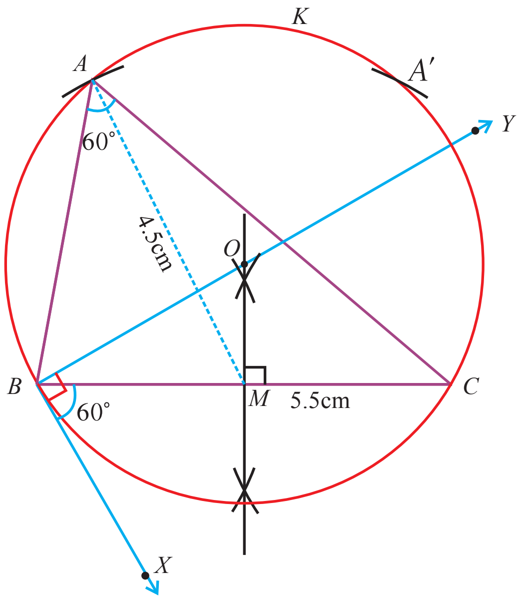
Construct a ΔABC in which $BC = 5.5$ cm., $\angle A = 60^\circ$ and the median AM from the vertex A is 4.5 cm.

Given : In ΔABC, $BC = 5.5$ cm, $\angle A = 60^\circ$, Median $AM = 4.5$ cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a line segment $BC = 5.5$ cm.

(ii) Through B draw BX such that $\angle CBX = 60^\circ$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC intersecting BY at O and BC at M.

(v) With O as centre and OB as radius, draw the circle.

(vi) The major arc BKC of the circle, contains the vertical angle 60°.

(vii) With M as centre, draw an arc of radius 4.5 cm meeting the circle at A and A'.

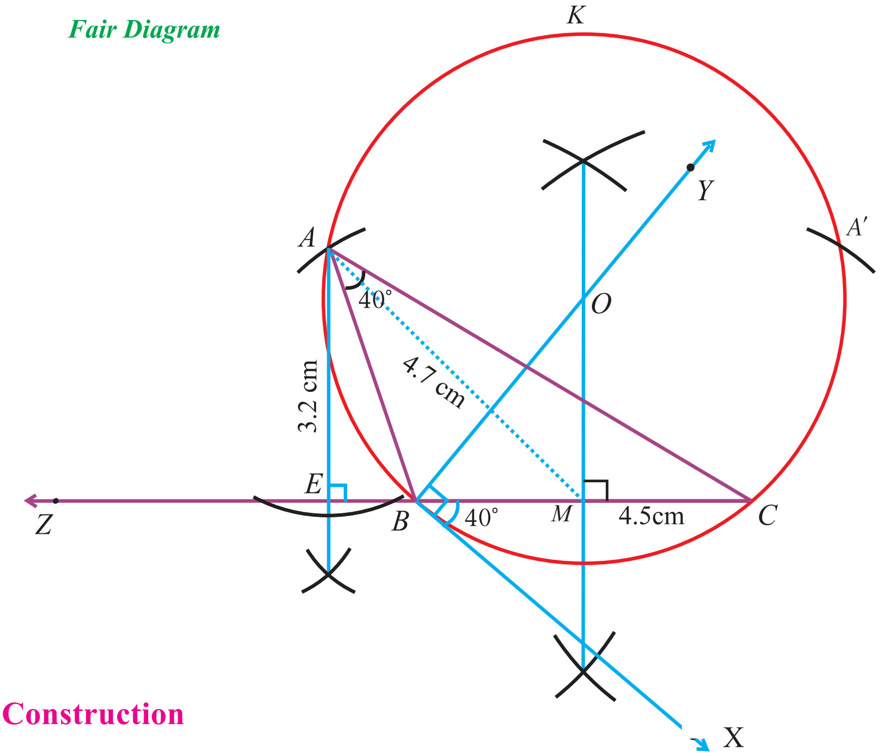
(viii) $\triangle ABC$ or $\Delta A'BC$ is the required triangle.

Example 9.6

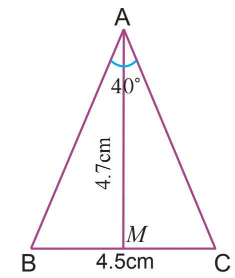
Construct a ΔABC, in which $BC = 4.5$ cm, $\angle A = 40^\circ$ and the median AM from A to BC is 4.7 cm. Find the length of the altitude from A to BC.

Given : In ΔABC, $BC = 4.5$ cm, $\angle A = 40^\circ$ and the median AM from A to BC is 4.7cm.

Fair Diagram



Rough Diagram



Construction

(i) Draw a line segment $BC = 4.5$cm.

(ii) Draw BX such that $\angle CBX = 40^\circ$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC intersecting BY at O and BC at M.

(v) With O as centre and OB as radius, draw the circle .

(vi) The major arc BKC of the circle, contains the vertical angle 40°.

(vii) With M as centre draw an arc of radius 4.7cm meeting the circle at A and A'.

(viii) Complete ΔABC or $\Delta A'BC$, which is the required triangle.

(ix) Produce CB to CZ.

(x) Draw $AE \perp CZ$.

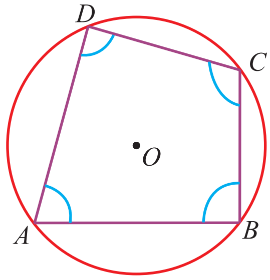
(xi) Length of the altitude AE is 3.2cm.

Exercise 9.2

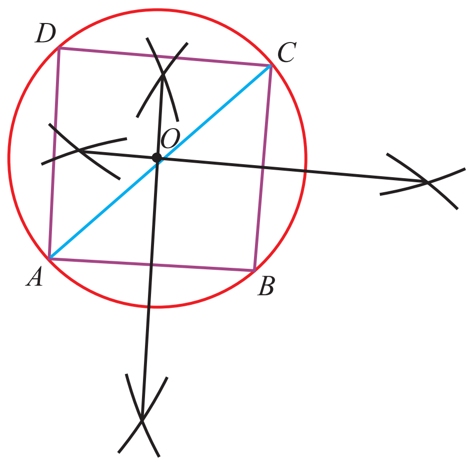
1. Construct a segment of a circle on a given line segment $AB = 5.2$ cm containing an angle 48°.
2. Construct a ΔPQR in which the base $PQ = 6$ cm, $\angle R = 60^\circ$ and the altitude from R to PQ is 4 cm.
3. Construct a ΔPQR such that $PQ = 4$ cm, $\angle R = 25^\circ$ and the altitude from R to PQ is 4.5 cm.
4. Construct a ΔABC such that $BC = 5$ cm. $\angle A = 45^\circ$ and the median from A to BC is 4cm.
5. Construct a ΔABC in which the base $BC = 5$ cm, $\angle BAC = 40^\circ$ and the median from A to BC is 6 cm. Also, measure the length of the altitude from A.

9.4 Construction of cyclic quadrilateral

If the vertices of a quadrilateral lie on a circle, then the quadrilateral is known as a cyclic quadrilateral. In a cyclic quadrilateral, the opposite angles are supplementary. That is, the sum of opposite angles is 180°. Thus, four suitable measurements (instead of five measurements) are sufficient for the construction of a cyclic quadrilateral.



Let us describe the various steps involved in the construction of a cyclic quadrilateral when the required measurements are given.



(i) Draw a rough figure and draw a $\triangle ABC$ or $\triangle ABD$ using the given measurements.

(ii) Draw the perpendicular bisectors of AB and BC intersecting each other at O. (one can take any two sides of $\triangle ABC$)

(iii) With O as the centre, and OA as radius, draw a circumcircle of $\triangle ABC$.

(iv) Using the given measurement, find the fourth vertex D and join AD and CD.

(v) Now, $ABCD$ is the required cyclic quadrilateral.

In this section, we shall construct a cyclic quadrilateral based on the different set of measurements of the cyclic quadrilateral as listed below.

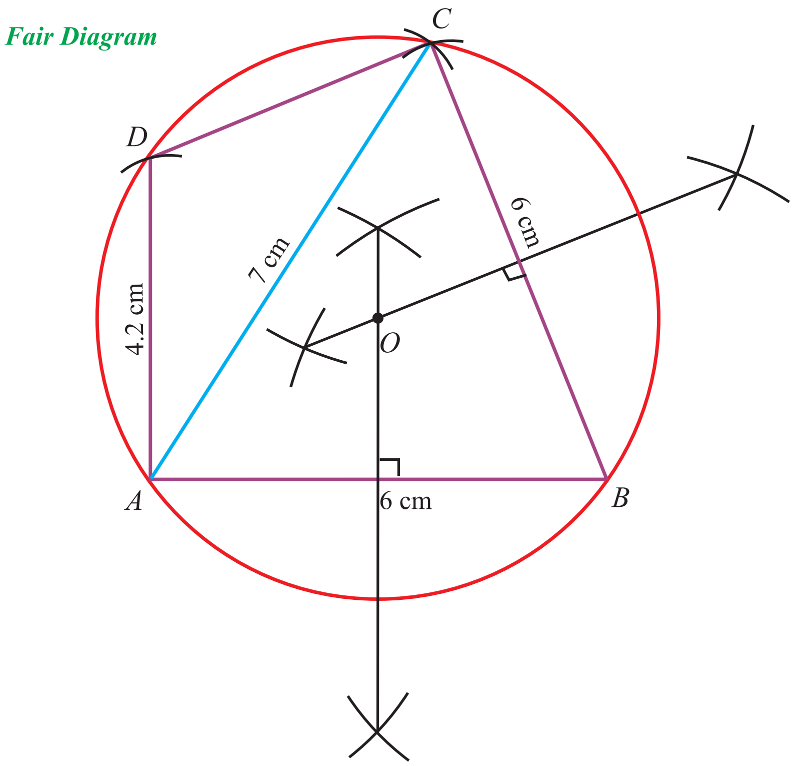
(i) Three sides and one diagonal. (ii) Two sides and two diagonals. (iii) Three sides and one angle. (iv) Two sides and two angles. (v) One side and three angles. (vi) Two sides, one angle and one parallel line.

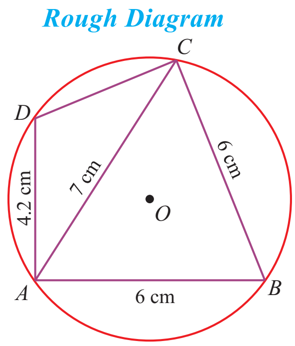
Type I (Three sides and one diagonal of a cyclic quadrilateral are given)

Example 9.7

Construct a cyclic quadrilateral $ABCD$ in which $AB = 6$ cm, $AC = 7$ cm, $BC = 6$ cm, and $AD = 4.2$ cm.

Given : In the cyclic quadrilateral $ABCD$, $AB = 6$ cm, $AC = 7$ cm. $BC = 6$ cm, and $AD = 4.2$ cm.





Construction

(i) Draw a rough diagram and mark the measurements.
Draw a line segment $AB = 6$ cm.

(ii) With A and B as centres, draw arcs with radii 7 cm and 6 cm respectively, to intersect at C. Join AC and BC.

(iii) Draw the perpendicular bisectors of AB and BC to intersect at O.

(iv) With O as the centre and OA $(= OB = OC)$ as radius draw the circumcircle of ΔABC

(v) With A as the centre and radius 4.2 cm. draw an arc intersecting the circumcircle at D.

(vi) Join AD and CD.
Now, $ABCD$ is the required cyclic quadrilateral.

Type II (Two sides and two diagonals of a cyclic quadrilateral are given)

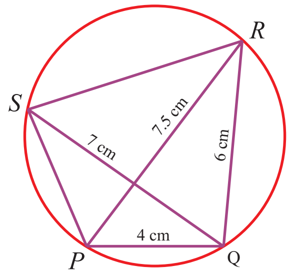
Example 9.8

Construct a cyclic quadrilateral $PQRS$ with $PQ = 4$ cm, $QR = 6$ cm, $PR = 7.5$ cm, $QS = 7$ cm

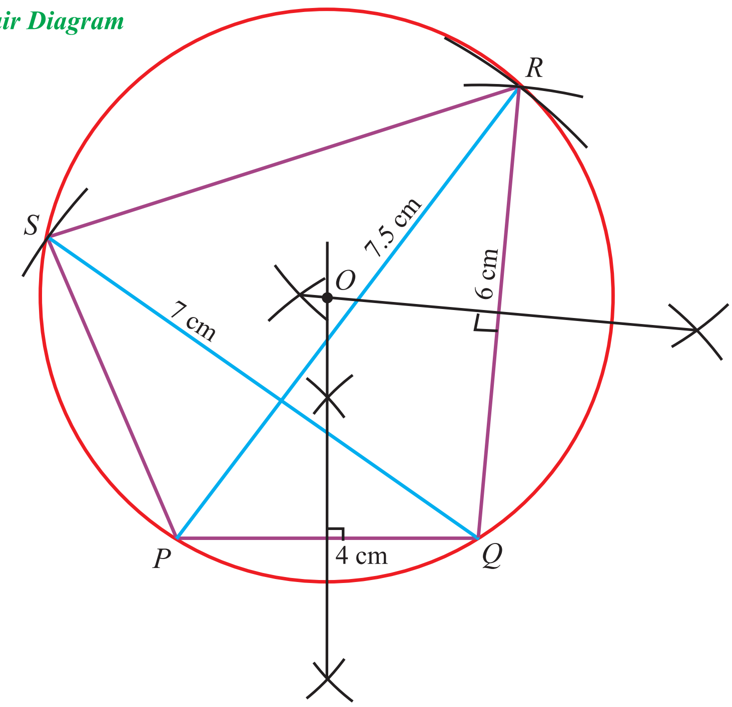
Given : In the cyclic quadrilateral $PQRS$, $PQ = 4$ cm, $QR = 6$ cm,

$PR = 7.5$ cm and $QS = 7$ cm

Rough diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.
Draw a line segment $PQ = 4$ cm

(ii) With P as centre and radius 7.5 cm, draw an arc.

(iii) With Q as centre and radius 6 cm, draw another arc meeting the previous arc as in the figure at R.

(iv) Join PR and QR.

(v) Draw the perpendicular bisectors of PQ and QR intersecting each other at O.

(vi) With O as the centre $OP(=OQ=OR)$ as radius, draw the circumcircle of ΔPQR.

(vii) With Q as centre and 7 cm radius , draw an arc intersecting the circle at S.

(viii) Join PS and RS.

(ix) Now, $PQRS$ is the required cyclic quadrilateral.

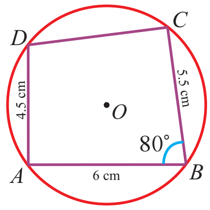
Type III (Three sides and one angle of a cyclic quadrilateral are given)

Example 9.9

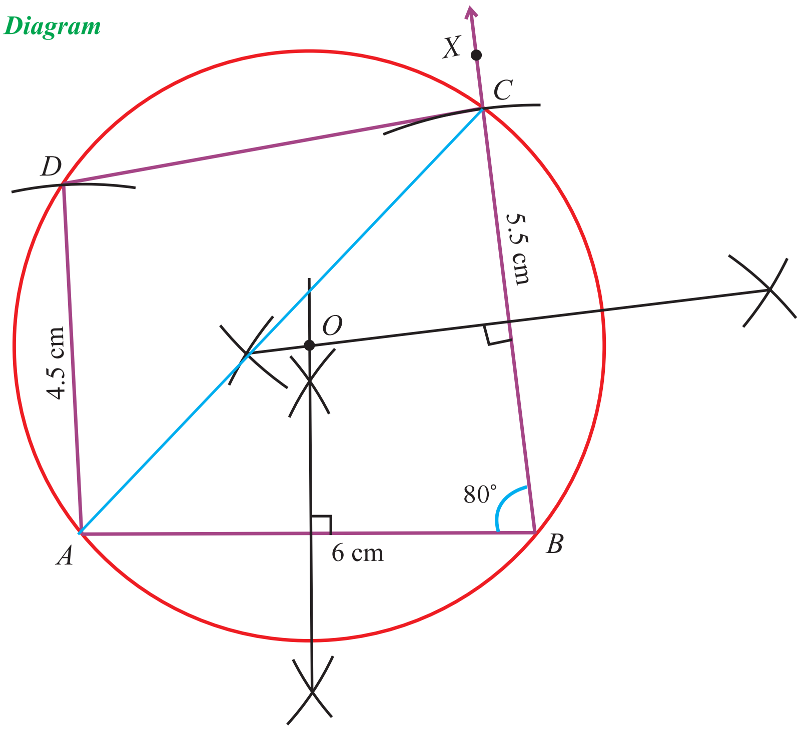
Construct a cyclic quadrilateral $ABCD$ when $AB = 6$ cm, $BC = 5.5$ cm, $\angle ABC = 80^\circ$ and $AD = 4.5$ cm.

Given: In the Cyclic Quadrilateral $ABCD$, $AB = 6$ cm, $BC = 5.5$ cm, $\angle ABC = 80^\circ$ and $AD = 4.5$ cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $AB = 6$ cm.

(ii) Through B draw BX such that $\angle ABX = 80^\circ$.

(iii) With B as centre and radius 5.5 cm, draw an arc intersecting BX at C and join AC.

(iv) Draw the perpendicular bisectors of AB and BC intersecting each other at O.

(v) With O as centre and OA ($= OB = OC$) as radius, draw the circumcircle of ΔABC.

(vi) With A as centre and radius 4.5 cm, draw an arc intersecting the circle at D.

(vii) Join AD and CD.

(viii) Now, $ABCD$ is the required cyclic quadrilateral.

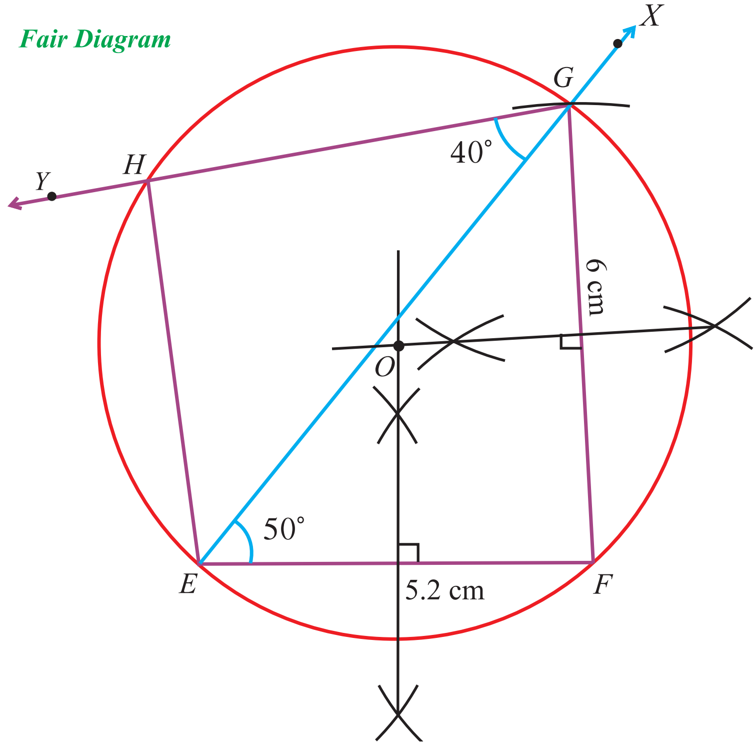
Type IV (Two sides and two angles of a cyclic quadrilateral are given)

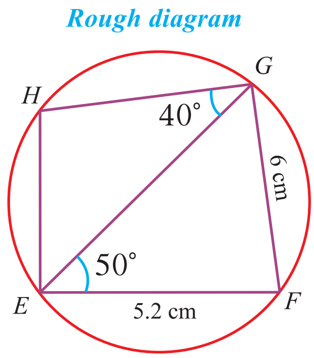
Example 9.10

Construct a cyclic quadrilateral $EFGH$ with $EF = 5.2$ cm, $\angle GEF = 50^\circ$, $FG = 6$ cm and $\angle EGH = 40^\circ$.

Given: In the Cyclic Quadrilateral $EFGH$

$EF = 5.2$ cm, $\angle GEF = 50^\circ$, $FG = 6$ cm and $\angle EGH = 40^\circ$.





Construction

(i) Draw a rough diagram and mark the measurements. Draw a line segment $EF = 5.2$ cm.

(ii) From E, draw EX such that $\angle FEX = 50^\circ$.

(iii) With F as centre and radius 6 cm, draw an arc intersecting EX at G.

(iv) Join FG.

(v) Draw the perpendicular bisectors of EF and FG intersecting each other at O.

(vi) With O as centre and OE $(= OF = OG)$ as radius, draw a circumcircle.

(vii) From G, draw GY such that $\angle EGY = 40^\circ$ which intersects the circle at H.

(viii) Join EH.

Now, $EFGH$ is the required cyclic quadrilateral.

Type V (One side and three angles of a cyclic quadrilateral are given)

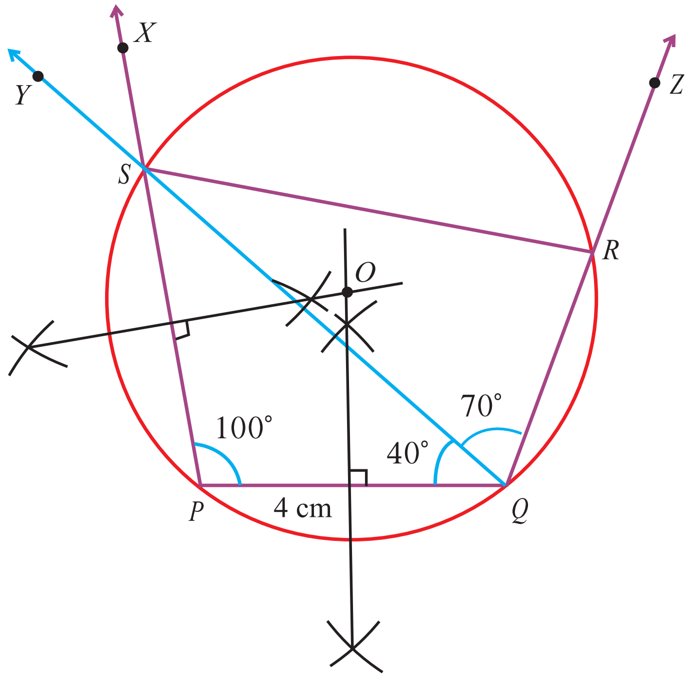
Example 9.11

Construct a cyclic quadrilateral $PQRS$ with $PQ = 4$ cm, $\angle P = 100^\circ$, $\angle PQS = 40^\circ$ and $\angle SQR = 70^\circ$.

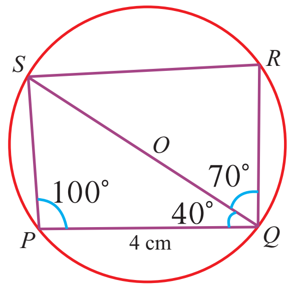
Given: In the cyclic quadrilateral $PQRS$,

$$PQ = 4\,\text{cm},\ \angle P = 100^\circ,\ \angle PQS = 40^\circ \text{ and } \angle SQR = 70^\circ.$$

Fair Diagram



Rough diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $PQ = 4$ cm.

(ii) From P draw PX such that $\angle QPX = 100^\circ$.

(iii) From Q draw QY such that $\angle PQY = 40^\circ$. Let QY meet PX at S.

(iv) Draw perpendicular bisectors of PQ and PS intersecting each other at O.

(v) With O as centre and $OP(= OQ = OS)$ as radius, draw a cicumcircle of $\triangle PQS$

(vi) From Q, draw QZ such that $\angle SQZ = 70^\circ$ which intersects the circle at R.

(vii) Join RS.

Now, $PQRS$ is the required cyclic quadrilateral.

Type VI

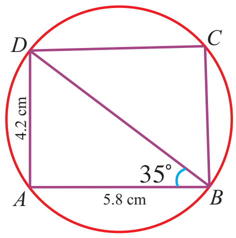
(Two sides , one angle and one parallel line are given)

Example 9.12

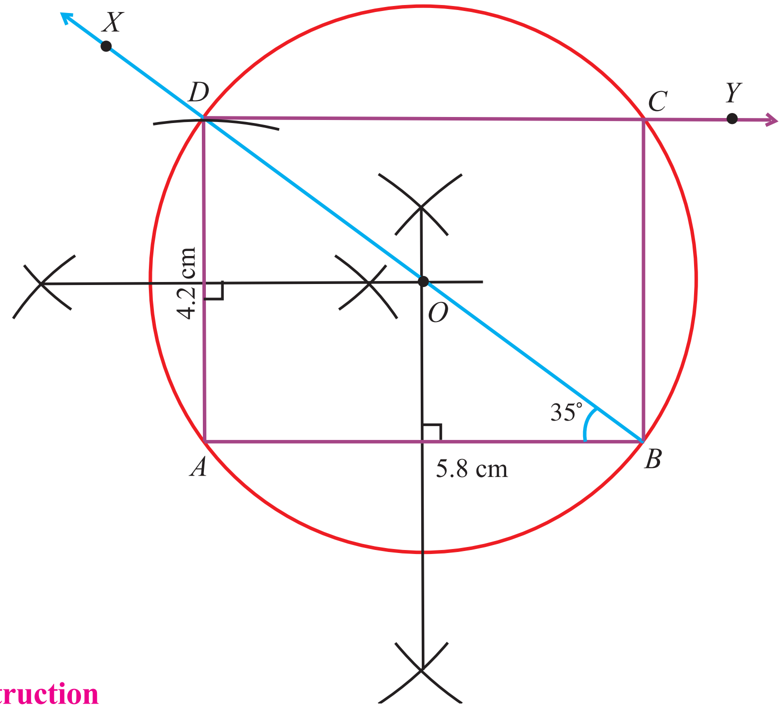
Construct a cyclic quadrilateral $ABCD$ when $AB = 5.8$ cm, $\angle ABD = 35^\circ$, $AD = 4.2$ cm and $AB \parallel CD$.

Given: In the cyclic quadrilateral $ABCD$, $AB = 5.8$ cm, $\angle ABD = 35^\circ$, $AD = 4.2$ cm and $AB \parallel CD$

Rough Diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $AB = 5.8$ cm.

(ii) From B, draw BX such that $\angle ABX = 35^\circ$.

(iii) With A as centre and radius 4.2 cm, draw an arc intersecting BX at D.

(iv) Draw perpendicular bisectors of AB and AD intersecting each other at O.

(v) With O as centre, and $OA\ (= OB = OD)$ as radius, draw a circumcircle of ΔABD.

(vi) Draw DY such that $DY \parallel AB$ intersecting the circle at C.

Join BC.

(vii) Now, $ABCD$ is the required cyclic quadrilateral.

Exercise 9.3

1. Construct a cyclic quadrilateral $PQRS$, with $PQ = 6.5$cm, $QR = 5.5$ cm, $PR = 7$cm and $PS = 4.5$cm.

2. Construct a cyclic quadrilateral $ABCD$ where $AB = 6$cm, AD = 4.8cm, $BD = 8$cm and $CD = 5.5$ cm.

3. Construct a cyclic quadrilateral $PQRS$ such that $PQ = 5.5$cm, $QR = 4.5$cm, $\angle QPR = 45^\circ$ and $PS = 3$cm.

4. Construct a cyclic quadrilateral $ABCD$ with $AB = 7$cm, $\angle A = 80^\circ$, $AD = 4.5$ cm and $BC = 5$ cm.

5. Construct a cyclic quadrilateral $KLMN$ such that $KL = 5.5$ cm, $KM = 5$ cm, $LM = 4.2$ cm and $LN = 5.3$ cm.

6. Construct a cyclic quadrilateral $EFGH$ where $EF = 7$ cm, $EH = 4.8$ cm, $FH = 6.5$ cm and $EG = 6.6$ cm.

7. Construct a cyclic quadrilateral $ABCD$, given $AB = 6$ cm, $\angle ABC = 70^\circ$, $BC = 5$cm and $\angle ACD = 30^\circ$

8. Construct a cyclic quadrilateral $PQRS$ given $PQ = 5$ cm, $QR = 4$ cm, $\angle QPR = 35^\circ$ and $\angle PRS = 70^\circ$

9. Construct a cyclic quadrilateral $ABCD$ such that $AB = 5.5$ cm $\angle ABC = 50^\circ$, $\angle BAC = 60^\circ$ and $\angle ACD = 30^\circ$

10. Construct a cyclic quadrilateral $ABCD$, where $AB = 6.5$ cm, $\angle ABC = 110^\circ$, $BC = 5.5$ cm and $AB \parallel CD$.

Do you know?

Every year since 1901, the prestigious Nobel Prize has been awarded to individuals for achievements in Physics, Chemistry, Physiology or medicine, Literature and for Peace. The Nobel Prize is an international award administered by the Nobel Foundation in Stockholm, Sweden. There is no Nobel Prize for Mathematics.

The Fields medal is a prize awarded to two , three or four Mathematicians not over 40 years of age at each International congress of the International Mathematical Union (IMU), a meeting that takes place every four years.

The Fields medal is often described as the Nobel Prize for Mathematics.

10

GRAPHS

I think, therefore I am

- Rene Descartes

- Introduction
- Quadratic Graphs
- Special Graphs

Rene Descartes
(1596-1650)
France

Descartes devised the cartesian plane while he was in a hospital bed watching a fly buzzing around a corner of his room.

He created analytical geometry which paved the way of plotting graphs using coordinate axes.

10.1 Introduction

Graphs are diagrams that show information. They show how two different quantities are related to each other like weight is related to height. Sometimes algebra may be hard to visualize. Learning to show relationships between symbolic expressions and their graphs opens avenues to realize algebraic patterns.

Students should acquire the habit of drawing a reasonably accurate graph to illustrate a given problem under consideration. A carefully made graph not only serves to clarify the geometric interpretation of a problem but also may serve as a valuable check on the accuracy of the algebraic work. One should never forget that graphical results are at best only approximations, and of value only in proportion to the accuracy with which the graphs are drawn.

10.2 Quadratic Graphs

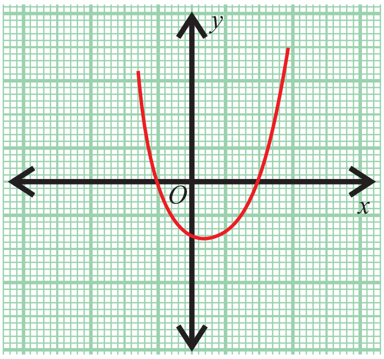
Definition

Let $f: A \to B$ be a function where A and B are subsets of $\mathbb{R}$. The set $\{(x,y) \mid x \in A, y = f(x)\}$ of all such ordered pairs (x, y) is called the graph of f.

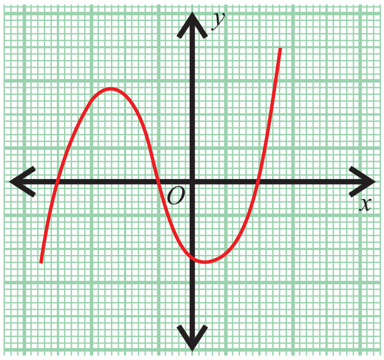
A polynomial function in x can be represented by a graph. The graph of a first degree polynomial $y = f(x) = ax + b, a \neq 0$ is an oblique line with slope a.

The graph of a second degree polynomial $y = f(x) = ax^2 + bx + c, \quad a \neq 0$ is a continuous non-linear curve, known as a parabola.

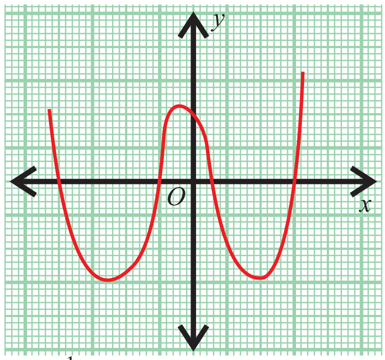
The following graphs represent different polynomials.



$y = (x+1)(x-2)$,
a polynomial of degree 2



$y = (x+4)(x+1)(x-2)$,
a polynomial of degree 3



$y = \frac{1}{14}(x+4)(x+1)(x-3)(x-0.5)$
a polynomial of degree 4

In class IX, we have learnt how to draw the graphs of linear polynomials of the form $y = ax + b, a \neq 0$. Now we shall focus on graphing a quadratic function $y = f(x) = ax^2 + bx + c$, where a, b and c are real constants, $a \neq 0$ and describe the nature of a quadratic graph.

Consider $y = ax^2 + bx + c$

By completing squares, the above polynomial can be rewritten as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

Hence $\frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right) \geq 0$. (square of an expression is always positive)

The vertex of the curve (parabola) is $V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

If $a > 0$, then the curve is open upward; it lies above or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

If $a < 0$, then the curve is open downward; it lies below or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

Let us give some examples of quadratic polynomials and the nature of their graphs in the following table.

S.No.	Polynomial ($y = ax^2 + bx + c$)	Vertex	Sign of a	Nature of curve
1	$y = 2x^2$ $a = 2,\ b = 0,\ c = 0$	(0, 0)	positive	(i) open upward (ii) lies above and on the line $y = 0$ (iii) symmetric about $x = 0$, i.e., y-axis
2	$y = -3x^2$ $a = -3,\ b = 0,\ c = 0$	(0, 0)	negative	(i) open downward (ii) lies below and on the line $y = 0$ (iii) symmetric about $x = 0$ i.e., y-axis
3	$y = x^2 - 2x - 3$ $a = 1, b = -2, c = -3$	(1, –4)	positive	(i) open upward (ii) lies above and on the line $y = -4$ (iii) symmetric about $x = 1$

Procedures to draw the quadratic graph $y = ax^2 + bx + c$

(i) Construct a table with the values of x and y using $y = ax^2 + bx + c$.

(ii) Choose a suitable scale.

The scale used on the x-axis does not have to be the same as the scale on the y-axis. The scale chosen should allow for the largest possible graph to be drawn. The bigger the graph, the more accurate will be the results obtained from it.

(iii) Plot the points on the graph paper and join these points by a smooth curve, as the graph of $y = ax^2 + bx + c$ does not contain line segments.

Example 10.1

Draw the graph of $y = 2x^2$.

Solution

First let us assign the integer values from -3 to 3 for x and form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

Plot the points $(-3, 18)$, $(-2, 8)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, $(2, 8)$, $(3, 18)$.

Join the points by a smooth curve.

The curve, thus obtained is the graph of $y = 2x^2$.

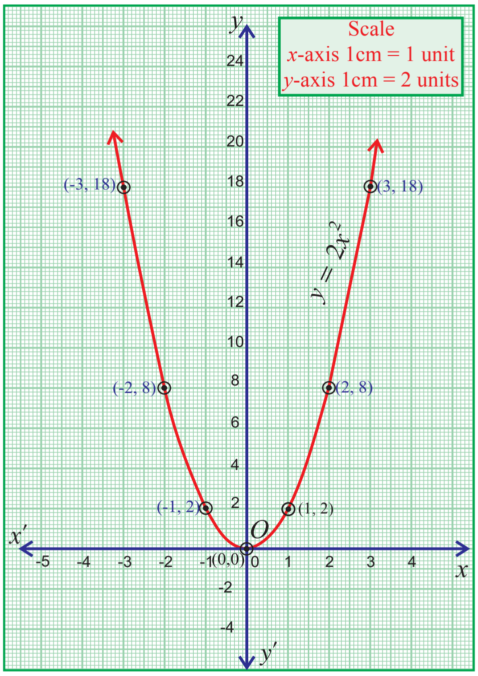


Fig. 10.1

Note

(i) It is symmetrical about y-axis. That is, the part of the graph to the left side of y-axis is the mirror image of the part to the right side of y-axis.

(ii) The graph does not lie below the x-axis as the values of y are non-negative.

Example 10.2

Draw the graph of $y = -3x^2$

Solution

Let us assign the integer values from -3 to 3 for x and form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = -3x^2$	-27	-12	-3	0	-3	-12	-27

Plot the points $(-3, -27)$, $(-2, -12)$, $(-1, -3)$, $(0, 0)$, $(1, -3)$, $(2, -12)$ and $(3, -27)$.

Join the points by a smooth curve.

The curve thus obtained, is the graph of $y = -3x^2$

Note

(i) The graph of $y = -3x^2$ does not lie above the x-axis as y is always negative.

(ii) The graph is symmetrical about y-axis.

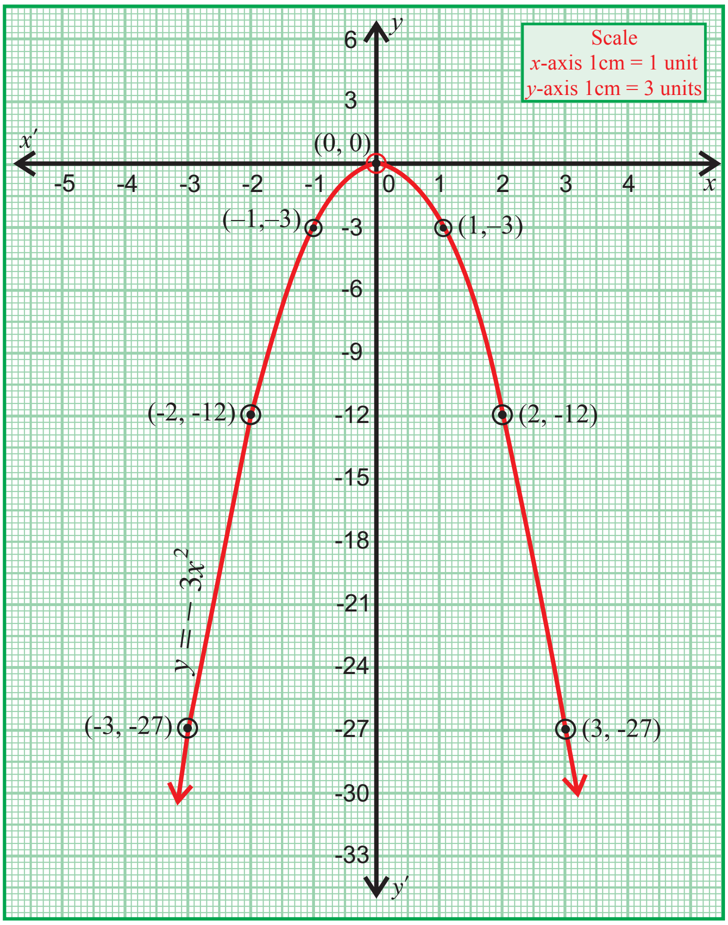


Fig. 10.2

10.2.1 To solve the quadratic equation $ax^2 + bx + c = 0$ graphically.

To find the roots of the quadratic equation $ax^2 + bx + c = 0$ graphically, let us draw the graph of $y = ax^2 + bx + c$. The x- coordinates of the points of intersection of the curve with the x-axis are the roots of the given equation, provided they intersect.

Example 10.3

Solve the equation $x^2 - 2x - 3 = 0$ graphically.

Solution

Let us draw the graph of $y = x^2 - 2x - 3$.

Now, form the following table by assigning integer values from –3 to 4 for x and finding the corresponding values of $y = x^2 - 2x - 3$.

x	–3	–2	–1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$-2x$	6	4	2	0	–2	–4	–6	–8
–3	–3	–3	–3	–3	–3	–3	–3	–3
y	12	5	0	–3	–4	–3	0	5

Plot the points (–3, 12), (–2, 5), (–1, 0), (0, –3), (1, –4), (2, –3), (3, 0), (4, 5) and join the points by a smooth curve.

The curve intersects the x-axis at the points (–1, 0) and (3, 0).

The x-coordinates of the above points are –1 and 3.

Hence, the solution set is {–1, 3}.

Note

(i) On the x-axis, $y = 0$ always.

(ii) The values of y are both positive and negative. Thus, the curve lies below and above the x-axis.

(iii) The curve is symmetric about the line $x = 1$. (It is not symmetric about the y-axis.)

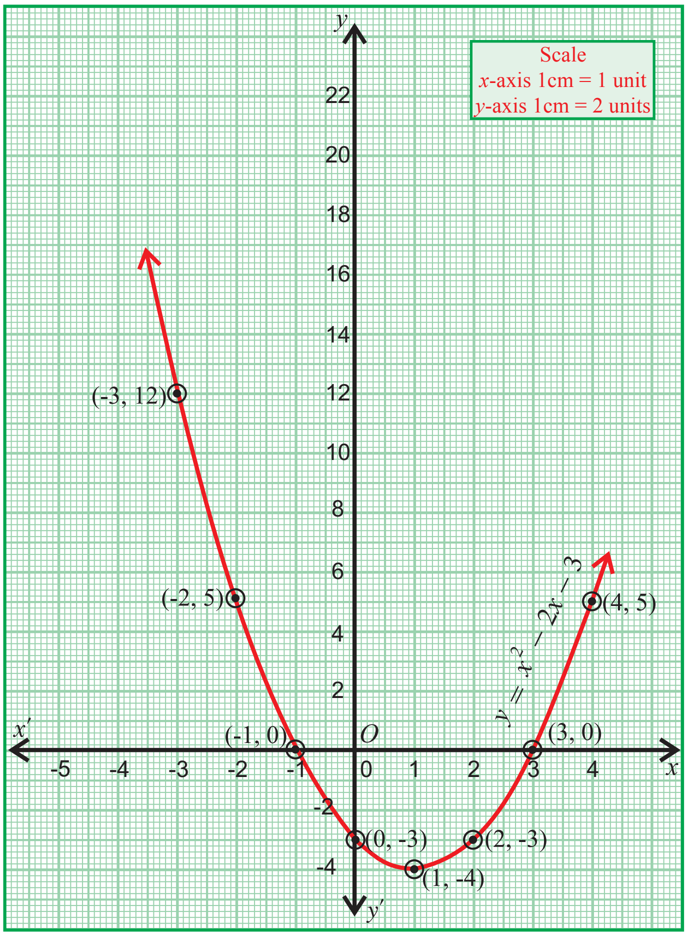


Fig. 10.3

Example 10.4

Solve graphically $2x^2 + x - 6 = 0$

Solution

First, let us form the following table by assigning integer values for x from –3 to 3 and finding the corresponding values of $y = 2x^2 + x - 6$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
x	–3	–2	–1	0	1	2	3
–6	–6	–6	–6	–6	–6	–6	–6
y	9	0	–5	–6	–3	4	15

Plot the points (–3, 9), (–2, 0), (–1, –5), (0, –6), (1, –3), (2, 4) and (3, 15) on the graph.

Join the points by a smooth curve. The curve, thus obtained, is the graph of $y = 2x^2 + x - 6$.

The curve cuts the x-axis at the points (–2 , 0) and (1.5 , 0).

The x-coordinates of the above points are –2 and 1.5.

Hence, the solution set is {–2 , 1.5 }.

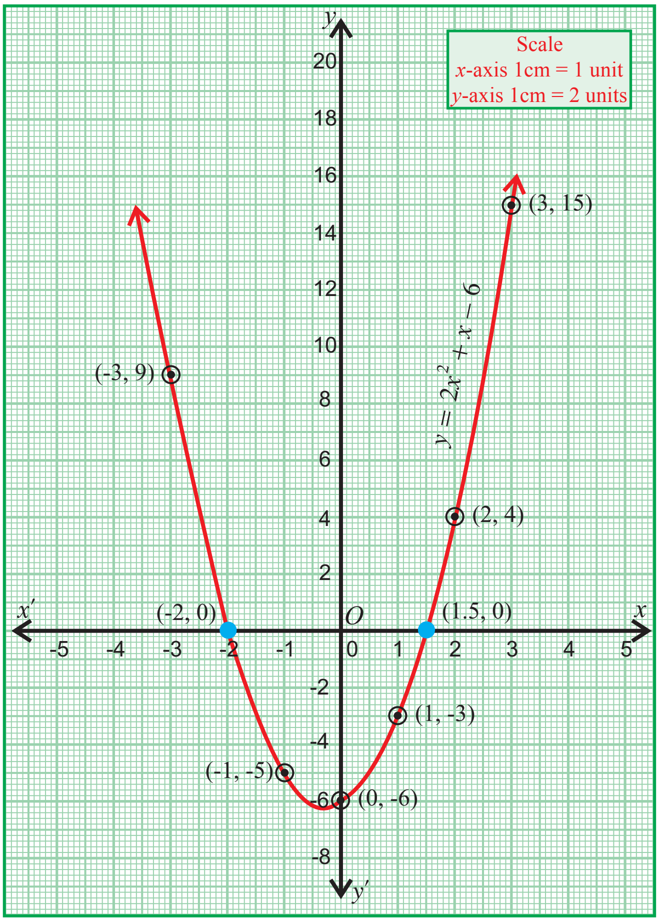


Fig. 10.4

Remarks

To solve $y = 2x^2 + x - 6$ graphically, one can proceed as follows.

(i) Draw the graph of $y = 2x^2$

(ii) Draw the graph of $y = 6 - x$

(iii) The x-coordinates of the points of intersection of the two graphs are the solutions of $2x^2 + x - 6 = 0$.

Example 10.5

Draw the graph of $y = 2x^2$ and hence solve $2x^2 + x - 6 = 0$.

Solution

First, let us draw the graph of $y = 2x^2$. Form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

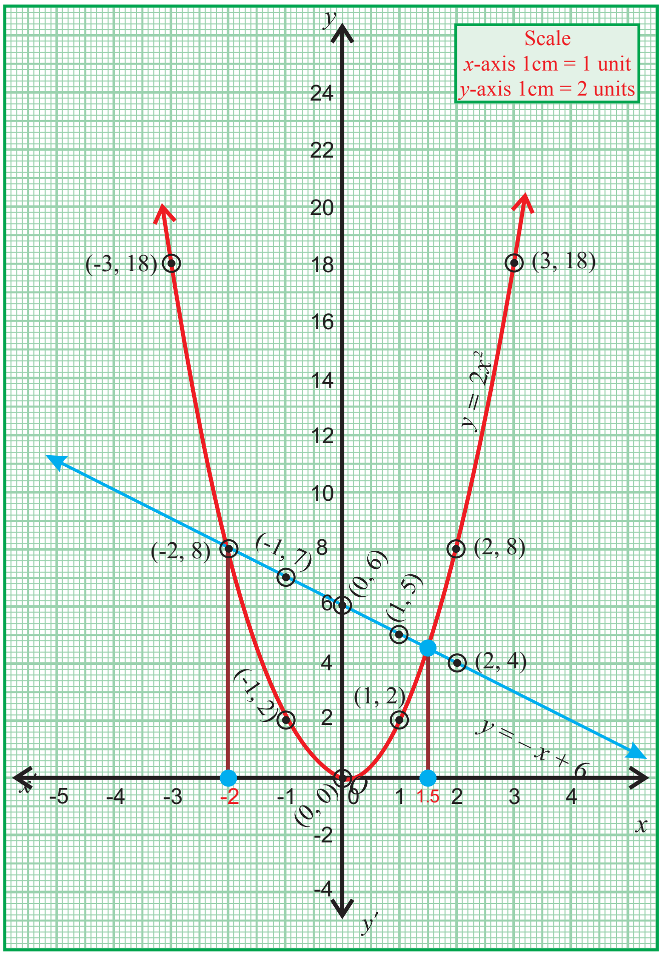


Fig. 10.5

Plot the points $(-3, 18)$, $(-2, 8)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, $(2, 8)$, $(3, 18)$.

Draw the graph by joining the points by a smooth curve.

To find the roots of $2x^2 + x - 6 = 0$, solve the two equations

$y = 2x^2$ and $2x^2 + x - 6 = 0$.

Now, $2x^2 + x - 6 = 0$.

$\Longrightarrow \quad y + x - 6 = 0$, since $y = 2x^2$

Thus, $y = -x + 6$

Hence, the roots of $2x^2 + x - 6 = 0$ are nothing but the x-coordinates of the points of intersection of

$y = 2x^2$ and $y = -x + 6$.

Now, for the straight line $y = -x + 6$, form the following table.

x	-1	0	1	2
$y = -x + 6$	7	6	5	4

Draw the straight line by joining the above points.

The points of intersection of the line and the parabola are $(-2, 8)$ and $(1.5, 4.5)$. The x-coordinates of the points are -2 and 1.5.

Thus, the solution set for the equation $2x^2 + x - 6 = 0$ is $\{-2, 1.5\}$.

Example 10.6

Draw the graph of $y = x^2 + 3x + 2$ and use it to solve the equation $x^2 + 2x + 4 = 0$.

Solution

First, let us form a table for $y = x^2 + 3x + 2$.

x	-4	-3	-2	-1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
$3x$	-12	-9	-6	-3	0	3	6	9
2	2	2	2	2	2	2	2	2
y	6	2	0	0	2	6	12	20

Plot the points $(-4, 6)$, $(-3, 2)$, $(-2, 0)$, $(-1, 0)$, $(0, 2)$, $(1, 6)$, $(2, 12)$ and $(3, 20)$.

Now, join the points by a smooth curve. The curve so obtained, is the graph of $y = x^2 + 3x + 2$.

Now, $x^2 + 2x + 4 = 0$

$\Longrightarrow \quad x^2 + 3x + 2 - x + 2 = 0$

$\Longrightarrow \quad y = x - 2 \qquad \because\ y = x^2 + 3x + 2$

Thus, the roots of $x^2 + 2x + 4 = 0$ are obtained form the points of intersection of

$y = x - 2$ and $y = x^2 + 3x + 2$.

Let us draw the graph of the straight line $y = x - 2$.

Now, form the table for the line $y = x - 2$

x	-2	0	1	2
$y = x - 2$	-4	-2	-1	0

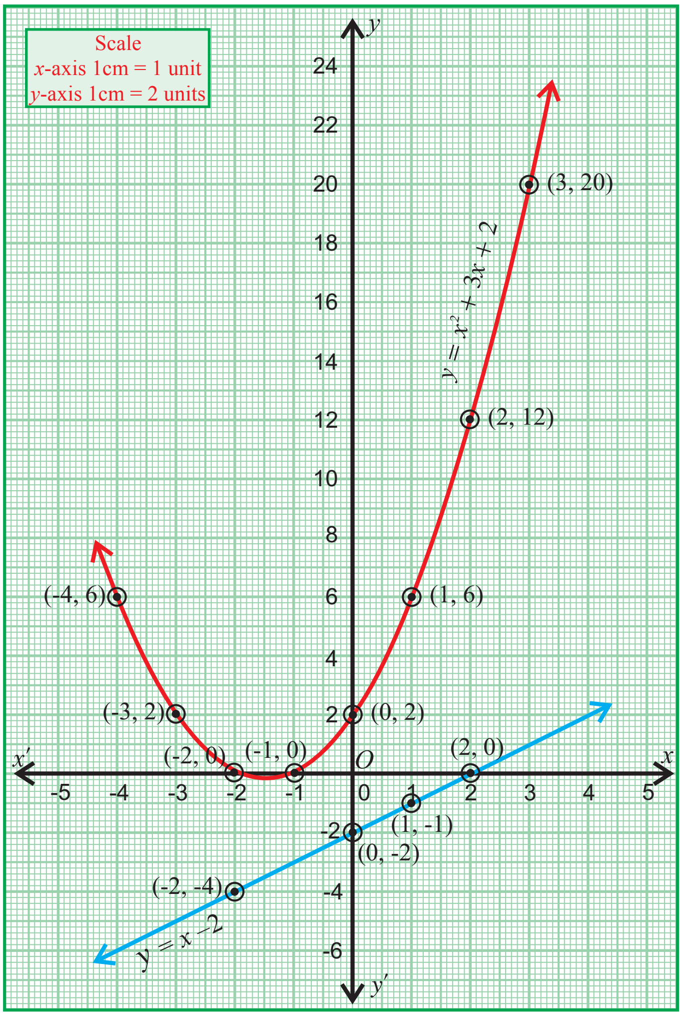


Fig. 10.6

The straight line $y = x - 2$ does not intersect the curve $y = x^2 + 3x + 2$.

Thus, $x^2 + 2x + 4 = 0$ has no real roots.

Exercise 10.1

1. Draw the graph of the following functions.

   (i) $y = 3x^2$

   (ii) $y = -4x^2$

   (iii) $y = (x+2)(x+4)$

   (iv) $y = 2x^2 - x + 3$

2. Solve the following equations graphically

   (i) $x^2 - 4 = 0$

   (ii) $x^2 - 3x - 10 = 0$

   (iii) $(x-5)(x-1) = 0$

   (iv) $(2x+1)(x-3) = 0$

3. Draw the graph of $y = x^2$ and hence solve $x^2 - 4x - 5 = 0$.

4. Draw the graph of $y = x^2 + 2x - 3$ and hence find the roots of $x^2 - x - 6 = 0$.

5. Draw the graph of $y = 2x^2 + x - 6$ and hence solve $2x^2 + x - 10 = 0$.

6. Draw the graph of $y = x^2 - x - 8$ and hence find the roots of $x^2 - 2x - 15 = 0$.

7. Draw the graph of $y = x^2 + x - 12$ and hence solve $x^2 + 2x + 2 = 0$.

10.3 Some Special Graphs

In this section, we will know how to draw graphs when the variables are in

(i) Direct variation (ii) Indirect variation.

If y is directly proportional to x, then we have $y = kx$, for some positive k. In this case the variables are said to be in direct variation and the graph is a straight line.

If y is inversely proportional to x, then we have $y = \frac{k}{x}$, for some positive k. In this case, the variables are said to be in indirect variation and the graph is a smooth curve , known as a Rectangular Hyperbola. (The equation of a rectangular hyperbola is of the form $xy = k$, $k > 0$.)

Example 10.7

Draw a graph for the following table and identify the variation.

x	2	3	5	8	10
y	8	12	20	32	40

Hence, find the value of y when $x = 4$.

Solution

From the table, we found that as x increases, y also increases. Thus, the variation is a direct variation.

Let $y = kx$.

$\Longrightarrow \quad \frac{y}{x} = k$

where k is the constant of proportionality.

From the given values, we have

$$k = \frac{8}{2} = \frac{12}{3} = \cdots = \frac{40}{10}. \quad \therefore\ k = 4$$

The relation $y = 4x$ forms a straight line graph.

Plot the points (2, 8), (3, 12), (5, 20), (8, 32) and (10, 40) and join these points to get the straight line.

Clearly, $y = 4x = 16$ when $x = 4$.

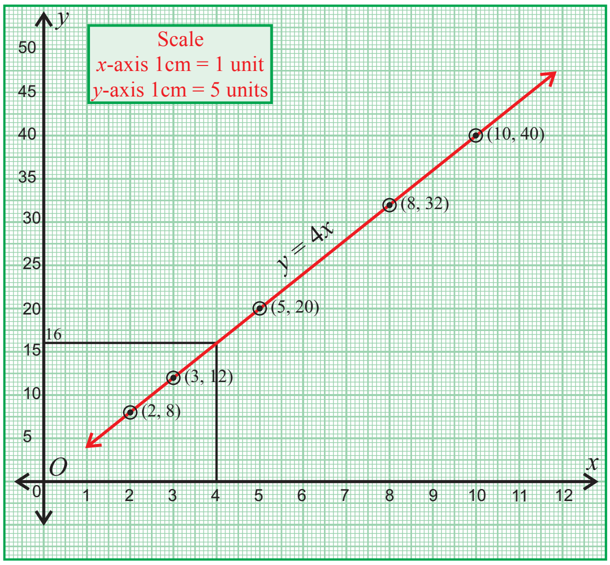


Fig. 10.7

Example 10.8

A cyclist travels from a place A to a place B along the same route at a uniform speed on different days. The following table gives the speed of his travel and the corresponding time he took to cover the distance.

Speed in km / hr x	2	4	6	10	12
Time in hrs y	60	30	20	12	10

Draw the speed-time graph and use it to find

(i) the number of hours he will take if he travels at a speed of 5 km / hr

(ii) the speed with which he should travel if he has to cover the distance in 40 hrs.

Solution

From the table, we observe that as x increases, y decreases.

This type of variation is called indirect variation.

Here, $xy = 120$.

Thus, $y = \dfrac{120}{x}$.

Plot the points (2 , 60), (4 , 30), (6 , 20), (10 , 12) and (12 , 10).

Join these points by a smooth curve.

From the graph, we have

(i) The number of hours he needed to travel at a speed of 5 km/hr is 24 hrs.

(ii) The required speed to cover the distance in 40 hrs, is 3 km / hr.

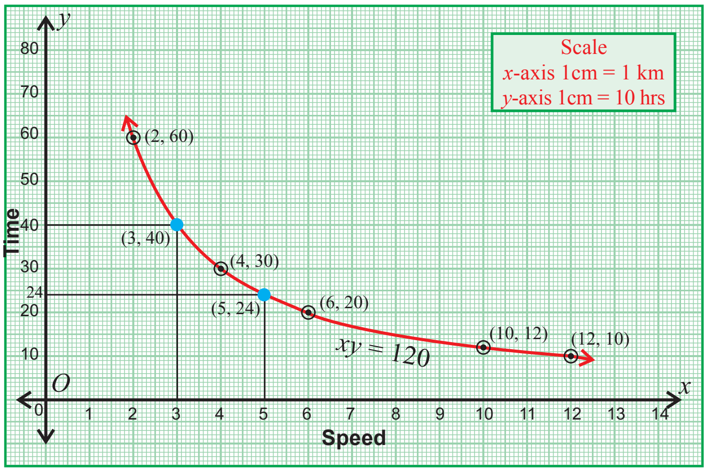


Fig. 10.8

Example 10.9

A bank gives 10% S.I on deposits for senior citizens. Draw the graph for the relation between the sum deposited and the interest earned for one year. Hence find

(i) the interest on the deposit of ₹650

(ii) the amount to be deposited to earn an interest of ₹45.

Solution

Let us form the following table.

Deposit ₹ x	100	200	300	400	500	600	700
S.I. earned ₹ y	10	20	30	40	50	60	70

Clearly $y = \dfrac{1}{10}x$ and the graph is a straight line.

Draw the graph using the points given in the table. From the graph, we see that

(i) The interest for the deposit of ₹650 is ₹65.

(ii) The amount to be deposited to earn an interest of ₹45 is ₹450.

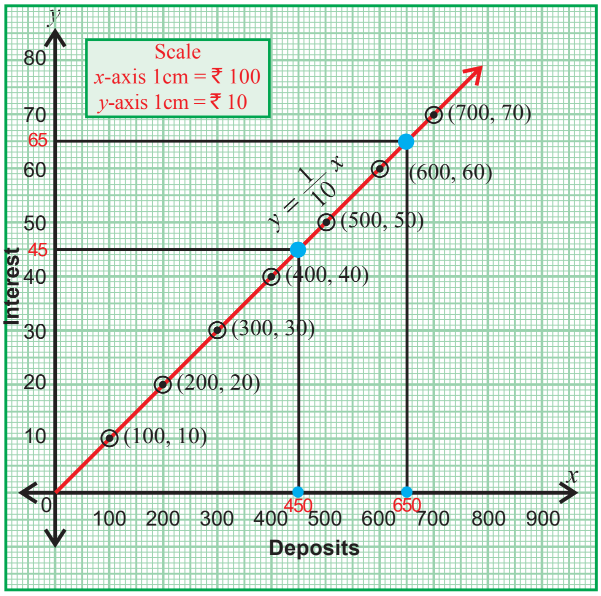


Fig. 10.9

Exercise 10.2

1. A bus travels at a speed of 40 km / hr. Write the distance-time formula and draw the graph of it. Hence, find the distance travelled in 3 hours.

2. The following table gives the cost and number of notebooks bought.

No. of note books x	2	4	6	8	10	12
Cost ₹ y	30	60	90	120	150	180

Draw the graph and hence (i) Find the cost of seven note books.

(ii) How many note books can be bought for ₹ 165.

3.

x	1	3	5	7	8
y	2	6	10	14	16

Draw the graph for the above table and hence find

(i) the value of y if $x = 4$

(ii) the value of x if $y = 12$

4. The cost of the milk per litre is ₹ 15. Draw the graph for the relation between the quantity and cost . Hence find

(i) the proportionality constant.

(ii) the cost of 3 litres of milk.

5. Draw the Graph of $xy = 20$, $x, y > 0$. Use the graph to find y when $x = 5$, and to find x when $y = 10$.

6.

No. of workers x	3	4	6	8	9	16
No of days y	96	72	48	36	32	18

Draw graph for the data given in the table. Hence find the number of days taken by 12 workers to complete the work.

Notable Quotes

1. In mathematics the art of proposing a question must be held of higher than solving it -Georg Cantor.
2. One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts - Albert Einstein

11

STATISTICS

It is easy to lie with statistics. It is hard to tell the truth without it
-Andrejs Dunkels

- Introduction
- Measures of Dispersion
    - Range
    - Variance
    - Standard Deviation
- Coefficient of Variation

Karl Pearson
(1857-1936)
England

Karl Pearson, British statistician, is a leading founder of modern field of statistics. He established the discipline of mathematical statistics. He introduced moments, a concept borrowed from physics.

His book, 'The Grammar of Science' covered several themes that were later to become part of the theories of Einstein and other scientists.

11.1 Introduction

According to Croxton and Cowden, Statistics is defined as the collection, presentation, analysis and interpretation of numerical data. R.A. Fisher said that the science of statistics is essentially a branch of Mathematics and may be regarded as mathematics applied to observational data. Horace Secrist defined statistics as follows:

"By statistics we mean aggregates of facts affected to a marked extent by multiplicity of causes, numerically expressed, enumerated or estimated according to reasonable standards of accuracy, collected in a systematic manner for a pre-determined purpose and placed in relation to each other".

The word 'Statistics' is known to have been used for the first time in " Elements of Universal Erudiation" by J.F. Baron. In modern times, statistics is no longer merely the collection of data and their presentation in charts and tables - it is now considered to encompass the science of basing inferences on observed data and the entire problem of making decisions in the face of uncertainity.

We have already learnt about the measures of central tendency namely, Mean, Median and Mode. They give us an idea of the concentration of the observation (data) about the central part of the distribution.

The knowledge of measures of central tendency cannot give a complete idea about the distribution. For example, consider the following two different series (i) 82, 74, 89, 95 and (ii) 120, 62, 28, 130. The two distributions have the same Mean 85. In the former, the numbers are closer to the mean 85 where as in the second series, the numbers are widely scattered about the Mean 85. Thus the measures of central tendency may mislead us. We need to have a measure which tells us how the items are dispersed around the Mean.

11.2 Measures of dispersion

Measures of dispersion give an idea about the scatteredness of the data of the distribution. Range (R), Quartile Deviation (Q.D), Mean Deviation (M.D) and Standard Deviation (S.D) are the measures of dispersion. Let us study about some of them in detail.

11.2.1 Range

Range is the simplest measure of dispersion. Range of a set of numbers is the difference between the largest and the smallest items of the set.

$\therefore$ Range = Largest value − Smallest value

$= L - S.$

The coefficient of range is given by $\dfrac{L-S}{L+S}$

Example 11.1

Find the range and the coefficient of range of 43, 24, 38, 56, 22, 39, 45.

Solution Let us arrange the given data in the ascending order.

22, 24, 38, 39, 43, 45, 56.

From the given data the largest value, $L = 56$ and the smallest value, $S = 22$.

$$\therefore \quad \text{Range} = L - S$$
$$= 56 - 22 = 34$$

Now the coefficient of range $= \dfrac{L-S}{L+S}$

$$= \frac{56-22}{56+22} = \frac{34}{78} = 0.436$$

Example 11.2

The weight (in kg) of 13 students in a class are 42.5, 47.5, 48.6, 50.5, 49, 46.2, 49.8, 45.8, 43.2, 48, 44.7, 46.9, 42.4. Find the range and coefficient of range.

Solution Let us arrange the given data in the ascending order.

42.4, 42.5, 43.2, 44.7, 45.8, 46.2, 46.9, 47.5, 48, 48.6, 49, 49.8, 50.5

From the given data, the largest value $L = 50.5$ and the smallest value $S = 42.4$

$$\text{Range} = L - S$$
$$= 50.5 - 42.4 = 8.1$$

The coefficient of range $= \dfrac{L-S}{L+S} = \dfrac{50.5-42.4}{50.5+42.4} = \dfrac{8.1}{92.9}$

$= 0.087$

Example 11.3

The largest value in a collection of data is 7.44. If the range is 2.26, then find the smallest value in the collection.

Solution Range = largest value − smallest value

$\Longrightarrow$ 7.44 − smallest value = 2.26

$\therefore$ The smallest value = 7.44 − 2.26 = 5.18

11.2.2 Standard deviation

A better way to measure dispersion is to square the differences between each data and the mean before averaging them. This measure of dispersion is known as the Variance and the positive square root of the Variance is known as the Standard Deviation. The variance is always positive.

The term 'standard deviation' was first used by Karl Pearson in 1894 as a replacement of the term 'mean error' used by Gauss.

Standard deviation is expressed in the same units as the data. It shows how much variation is there from the mean. A low standard deviation indicates that the data points tend to be very close to the mean, where as a high standard deviation indicates that the data is spread out over a large range of values.

We use $\overline{x}$ and σ to denote the mean and the standard deviation of a distribution respectively. Depending on the nature of data, we shall calculate the standard deviation σ (after arranging the given data either in ascending or descending order) by different methods using the following formulae (proofs are not given).

Data	Direct method	Actual mean method	Assumed mean method	Step deviation method
Ungrouped	$\sqrt{\frac{\sum x^2}{n}-\left(\frac{\sum x}{n}\right)^2}$	$\sqrt{\frac{\sum d^2}{n}}$ $d = x-\overline{x}$	$\sqrt{\frac{\sum d^2}{n}-\left(\frac{\sum d}{n}\right)^2}$ $d = x-A$	$\sqrt{\frac{\sum d^2}{n}-\left(\frac{\sum d}{n}\right)^2}\times c$ $d=\frac{x-A}{c}$
Grouped		$\sqrt{\frac{\sum fd^2}{\sum f}}$	$\sqrt{\frac{\sum fd^2}{\sum f}-\left(\frac{\sum fd}{\sum f}\right)^2}$	$\sqrt{\frac{\sum fd^2}{\sum f}-\left(\frac{\sum fd}{\sum f}\right)^2}\times c$

Note

For a collection of n items (numbers), we always have $\sum(x-\overline{x})=0$, $\sum x = n\overline{x}$ and $\sum \overline{x} = n\overline{x}$.

(i) Direct method

This method can be used, when the squares of the items are easily obtained.

Example 11.4

The number of books read by 8 students during a month are

2, 5, 8, 11, 14, 6, 12, 10. Calculate the standard deviation of the data.

Solution

x	x^2
2	4
5	25
6	36
8	64
10	100
11	121
12	144
14	196
$\sum x=68$	$\sum x^2=690$

Here, the number of items, $n = 8$

Standard deviation, $\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$

$= \sqrt{\frac{690}{8} - \left(\frac{68}{8}\right)^2}$

$= \sqrt{86.25 - (8.5)^2}$

$= \sqrt{86.25 - 72.25}$

$= \sqrt{14} \simeq 3.74$

(ii) Actual mean method

This method can be used when the mean is not a fraction.

The standard deviation, $\sigma = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$ or $\sigma = \sqrt{\frac{\sum d^2}{n}}$, where $d = x - \overline{x}$.

Example 11.5

A test in General Knowledge was conducted for a class. The marks out of 40, obtained by 6 students were 20, 14, 16, 30, 21 and 25. Find the standard deviation of the data.

Solution Now, A. M. $= \frac{\sum x}{n} = \frac{20+14+16+30+21+25}{6}$

$\Longrightarrow \quad \overline{x} = \frac{126}{6} = 21.$

Let us form the following table.

x	$d = x - \overline{x}$	d^2
14	-7	49
16	-5	25
20	-1	1
$\boxed{21}$	0	0
25	4	16
30	9	81
$\sum x = 126$	$\sum d = 0$	$\sum d^2 = 172$

$\sigma = \sqrt{\frac{\sum d^2}{n}} = \sqrt{\frac{172}{6}}$

$= \sqrt{28.67}$

Thus, $\sigma \simeq 5.36$

(iii) Assumed mean method

When the mean of the given data is not an integer, we use assumed mean method to calculate the standard deviation. We choose a suitable item A such that the difference $x–A$ are all small numbers possibly, integers. Here A is an assumed mean which is supposed to be closer to the mean.

We calculate the deviations using $d = x - A$.

Now the standard deviation, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$.

> **Note**
>
> Assumed mean method and step deviation method are just simplified forms of direct method.

Example: 11.6

Find the standard deviation of the numbers 62, 58, 53, 50, 63, 52, 55.

Solution Let us take $A = 55$ as the assumed mean and form the following table.

x	$d = x - A$ $= x - 55$	d^2
50	-5	25
52	-3	9
53	-2	4
55	0	0
58	3	9
62	7	49
63	8	64
	$\sum d = 8$	$\sum d^2 = 160$

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{160}{7} - \left(\frac{8}{7}\right)^2}$$

$$= \sqrt{\frac{160}{7} - \frac{64}{49}}$$

$$= \sqrt{\frac{1056}{49}}$$

$$= \frac{32.49}{7}$$

$\therefore$ Standard deviation $\sigma \simeq 4.64$

(iv) Step deviation method

This method can be used to find the standard deviation when the items are larger in size and have a common factor. We choose an assumed mean A and calculate d *by* using $d = \frac{x - A}{c}$ where c is the common factor of the values of $x–A$.

We use the formula, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times c$.

Example 11.7

The marks obtained by 10 students in a test in Mathematics are :

80, 70, 40, 50, 90, 60, 100, 60, 30, 80. Find the standard deviation.

Solution We observe that all the data have 10 as common factor. Take $A = 70$ as assumed mean. Here the number of items, $n = 10$.

Take $c = 10$, $d = \frac{x - A}{10}$ and form the following table.

x	$d = \frac{x - 70}{10}$	d^2
30	-4	16
40	-3	9
50	-2	4
60	-1	1
60	-1	1
$\boxed{70}$	0	0
80	1	1
80	1	1
90	2	4
100	3	9
	$\sum d = -4$	$\sum d^2 = 46$

Now $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times c$

$$= \sqrt{\frac{46}{10} - \left(\frac{-4}{10}\right)^2} \times 10$$

$$= \sqrt{\frac{46}{10} - \frac{16}{100}} \times 10 = \sqrt{\frac{460 - 16}{100}} \times 10$$

$\therefore$ Standard deviation, $\sigma \simeq 21.07$

The standard deviation for a collection of data can be obtained in any of the four methods, namely direct method, actual mean method, assumed mean method and step deviation method.

As expected, the different methods should not give different answers for σ for the same data. Students are advised to follow any one of the above methods.

Results

(i) The standard deviation of a distribution remains unchanged when each value is added or subtracted by the same quantity.

(ii) If each value of a collection of data is multiplied or divided by a non-zero constant k, then the standard deviation of the new data is obtained by multiplying or dividing the standard deviation by the same quantity k.

Example: 11.8

Find the standard deviation of the data 3, 5, 6, 7. Then add 4 to each item and find the standard deviation of the new data.

Solution Given data 3, 5, 6, 7

Take $A = 6$

x	$d = x - 6$	d^2
3	-3	9
5	-1	1
$\boxed{6}$	0	0
7	1	1
	$\Sigma d = -3$	$\Sigma d^2 = 11$

$$\text{Standard deviation}, \sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{11}{4} - \left(\frac{-3}{4}\right)^2}$$

$$\sigma = \sqrt{\frac{11}{4} - \frac{9}{16}} = \frac{\sqrt{35}}{4}$$

Let us add 4 to each term of the given data to get the new data 7, 9, 10, 11

Take $A = 10$

x	$d = x - 10$	d^2
7	-3	9
9	-1	1
$\boxed{10}$	0	0
11	1	1
	$\Sigma d = -3$	$\Sigma d^2 = 11$

$$\text{Standard deviation}, \sigma_1 = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{11}{4} - \left(\frac{-3}{4}\right)^2}$$

$$\sigma_1 = \sqrt{\frac{11}{4} - \frac{9}{16}} = \frac{\sqrt{35}}{4}$$

In the above example, the standard deviation remains unchanged even when each item is added by the constant 4.

Example 11.9

Find the standard deviation of 40, 42 and 48. If each value is multiplied by 3, find the standard deviation of the new data.

Solution Let us consider the given data 40, 42, 48 and find σ.

Let the assumed mean A be 44

x	$d = x - 44$	d^2
40	-4	16
42	-2	4
48	4	16
	$\Sigma d = -2$	$\Sigma d^2 = 36$

$$\text{Standard deviation}, \sigma = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{36}{3} - \left(\frac{-2}{3}\right)^2}$$

$$\sigma = \frac{\sqrt{104}}{3}$$

When the values are multiplied by 3, we get 120,126,144. Let the assumed mean A be 132.

Let σ_1 be the S.D. of the new data.

x	$d = x - 132$	d^2
120	-12	144
126	-6	36
144	12	144
	$\Sigma d = -6$	$\Sigma d^2 = 324$

$$\text{Standard deviation}, \sigma_1 = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{324}{3} - \left(\frac{-6}{3}\right)^2}$$

$$\sigma_1 = \sqrt{\frac{312}{3}} = \sqrt{104}$$

In the above example, when each value is multiplied by 3, the standard deviation also gets multiplied by 3 .

Example 11.10

Prove that the standard deviation of the first n natural numbers is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

Solution The first n natural numbers are $1, 2, 3, \cdots, n$.

$$\text{Their mean, } \quad \overline{x} = \frac{\Sigma x}{n} = \frac{1+2+3+\cdots+n}{n}$$

$$= \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

Sum of the squares of the first n natural numbers is

$$\sum x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus, the standard deviation $\sigma = \sqrt{\dfrac{\sum x^2}{n} - \left(\dfrac{\sum x}{n}\right)^2}$

$$= \sqrt{\frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2}$$

$$= \sqrt{\frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left[\frac{(2n+1)}{3} - \frac{(n+1)}{2}\right]}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left[\frac{2(2n+1)-3(n+1)}{6}\right]}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left(\frac{4n+2-3n-3}{6}\right)}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left(\frac{n-1}{6}\right)}$$

$$= \sqrt{\frac{n^2-1}{12}}.$$

Hence, the S.D. of the first n natural numbers is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

Remarks

It is quite interesting to note the following:

The S.D. of any n successive terms of an A.P. with common difference d is, $\sigma = d\sqrt{\dfrac{n^2-1}{12}}$.

Thus,

(i) S.D. of $i, i+1, i+2, \cdots, i+n$ is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$, $i \in \mathbb{N}$.

(ii) S.D. of any n consecutive even integers, is given by $\sigma = 2\sqrt{\dfrac{n^2-1}{12}}$, $n \in \mathbb{N}$.

(iii) S.D. of any n consecutive odd integers, is given by $\sigma = 2\sqrt{\dfrac{n^2-1}{12}}$, $n \in \mathbb{N}$.

Example 11.11

Find the standard deviation of the first 10 natural numbers.

Solution Standard deviation of the first n natural numbers = $\sqrt{\frac{n^2-1}{12}}$

Standard deviation of the first 10 natural numbers

$$= \sqrt{\frac{10^2-1}{12}} = \sqrt{\frac{100-1}{12}} \simeq 2.87$$

Standard Deviation of grouped data

(i) Actual mean method

In a discrete data, when the deviations are taken from arithmetic mean, the standard deviation can be calculated using the formula $\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$, where $d = x - \overline{x}$.

Example 11.12

The following table shows the marks obtained by 48 students in a Quiz competition in Mathematics. Calculate the standard deviation.

Data x	6	7	8	9	10	11	12
Frequency f	3	6	9	13	8	5	4

Solution Let us form the following table using the given data.

x	f	fx	$d = x - \overline{x}$ $= x - 9$	fd	fd^2
6	3	18	-3	-9	27
7	6	42	-2	-12	24
8	9	72	-1	-9	9
$\boxed{9}$	13	117	0	0	0
10	8	80	1	8	8
11	5	55	2	10	20
12	4	48	3	12	36
	$\sum f = 48$	$\sum fx = 432$	$\sum d = 0$	$\sum fd = 0$	$\sum fd^2 = 124$

Arithmetic mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{432}{48} = 9.$

Standard deviation, $\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$

$$= \sqrt{\frac{124}{48}}$$

$$\sigma = \sqrt{2.58} \simeq 1.61$$

(ii) Assumed mean method

When deviations are taken from the assumed mean, the formula for calculating standard deviation is

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}\text{, where } d = x - A.$$

Example 11.13

Find the standard deviation of the following distribution.

x	70	74	78	82	86	90
f	1	3	5	7	8	12

Solution Let us take the assumed mean $A = 82$.

x	f	$d = x - 82$	fd	fd^2
70	1	-12	-12	144
74	3	-8	-24	192
78	5	-4	-20	80
$\boxed{82}$	7	0	0	0
86	8	4	32	128
90	12	8	96	768
	$\sum f = 36$		$\sum fd = 72$	$\sum fd^2 = 1312$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

$$= \sqrt{\frac{1312}{36} - \left(\frac{72}{36}\right)^2}$$

$$= \sqrt{\frac{328}{9} - 2^2}$$

$$= \sqrt{\frac{328 - 36}{9}}$$

$$= \sqrt{\frac{292}{9}} = \sqrt{32.44}$$

$$\therefore \quad \sigma \simeq 5.7$$

Example 11.14

Find the variance of the following distribution.

Class interval	3.5-4.5	4.5-5.5	5.5-6.5	6.5-7.5	7.5-8.5
Frequency	9	14	22	11	17

Solution Let the assumed mean A be 6.

class interval	x mid value	f	$d = x - 6$	fd	fd^2
3.5-4.5	4	9	-2	-18	36
4.5-5.5	5	14	-1	-14	14
5.5-6.5	$\boxed{6}$	22	0	0	0
6.5-7.5	7	11	1	11	11
7.5-8.5	8	17	2	34	68
		$\sum f = 73$		$\sum fd = 13$	$\sum fd^2 = 129$

$$\text{Now variance, } \sigma^2 = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

$$= \frac{129}{73} - \left(\frac{13}{73}\right)^2 \quad = \frac{129}{73} - \frac{169}{5329}$$

$$= \frac{9417 - 169}{5329} \quad = \frac{9248}{5329}$$

Thus, the variance is $\sigma^2 \simeq 1.74$

(iii) Step deviation method

Example 11.15

The following table gives the number of goals scored by 71 leading players in International Football matches. Find the standard deviation of the data.

Class Interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	8	12	17	14	9	7	4

Solution Let $A = 35$. In the 4th column, the common factor of all items, $c = 10$.

class interval	x mid value	f	$x-A$	$d = \frac{x-A}{c}$	fd	fd^2
0–10	5	8	–30	-3	-24	72
10–20	15	12	–20	-2	-24	48
20–30	25	17	–10	-1	-17	17
30–40	$\boxed{35}$	14	0	0	0	0
40–50	45	9	10	1	9	9
50–60	55	7	20	2	14	28
60–70	65	4	30	3	12	36
		$\sum f = 71$			$\sum fd = -30$	$\sum fd^2 = 210$

Standard deviation, $\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$

$$= \sqrt{\frac{210}{71} - \left(\frac{-30}{71}\right)^2} \times 10$$

$$= \sqrt{\frac{210}{71} - \frac{900}{5041}} \times 10$$

$$= \sqrt{\frac{14910 - 900}{5041}} \times 10$$

$$= \sqrt{\frac{14010}{5041}} \times 10 = \sqrt{2.7792} \times 10$$

Standard deviation, $\sigma \simeq 16.67$

Example 11.16

Length of 40 bits of wire, correct to the nearest centimetre are given below. Calculate the variance.

Length cm	1-10	11-20	21-30	31-40	41-50	51-60	61-70
No. of bits	2	3	8	12	9	5	1

Solution Let the assumed mean A be 35.5

Length	mid value x	no. of bits (f)	$d = x - A$	fd	fd^2
1-10	5.5	2	-30	-60	1800
11-20	15.5	3	-20	-60	1200
21-30	25.5	8	-10	-80	800
31-40	$\boxed{35.5}$	12	0	0	0
41-50	45.5	9	10	90	900
51-60	55.5	5	20	100	2000
61-70	65.5	1	30	30	900
		$\sum f = 40$		$\sum fd = 20$	$\sum fd^2 = 7600$

Variance, $\sigma^2 = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 = \frac{7600}{40} - \left(\frac{20}{40}\right)^2$

$$= 190 - \frac{1}{4} = \frac{760 - 1}{4} = \frac{759}{4}$$

$\therefore \quad \sigma^2 = 189.75$

11.2.3 Coefficient of variation

Coefficient of variation is defined as

$$\text{C.V} = \frac{\sigma}{\overline{x}} \times 100$$

where σ is the standard deviation and $\overline{x}$ is the mean of the given data. It is also called as a relative standard deviation.

Remarks

(i) The coefficient of variation helps us to compare the consistency of two or more collections of data.

(ii) When the coefficient of variation is more, the given data is less consistent.

(iii) When the coefficient of variation is less, the given data is more consistent.

Example 11.17

Find the coefficient of variation of the following data. 18, 20, 15, 12, 25.

Solution Let us calculate the A.M of the given data.

$$\text{A.M } \overline{x} = \frac{12+15+18+20+25}{5}$$

$$= \frac{90}{5} = 18.$$

x	$d = x - 18$	d^2
12	-6	36
15	-3	9
$\boxed{18}$	0	0
20	2	4
25	7	49
	$\sum d = 0$	$\sum d^2 = 98$

$$\sigma = \sqrt{\frac{\sum d^2}{n}} = \sqrt{\frac{98}{5}}$$

$$= \sqrt{19.6} \simeq 4.428$$

$\therefore$ The coefficient of variation $= \frac{\sigma}{\overline{x}} \times 100$

$$= \frac{4.428}{18} \times 100 = \frac{442.8}{18}.$$

$\therefore$ The coefficient of variation is 24.6

Example 11.18

Following are the runs scored by two batsmen in 5 cricket matches.Who is more consistent in scoring runs.

Batsman A	38	47	34	18	33
Batsman B	37	35	41	27	35

Solution

Batsman A

x	$d = x - \bar{x}$	d^2
18	-16	256
33	-1	1
$\boxed{34}$	0	0
38	4	16
47	13	169
170	0	442

Now $\bar{x} = \dfrac{170}{5} = 34$

$$\sigma = \sqrt{\frac{\sum d^2}{n}}$$

$$= \sqrt{\frac{442}{5}} = \sqrt{88.4}$$

$$\simeq 9.4$$

Coefficient of variation, C.V $= \dfrac{\sigma}{\bar{x}} \times 100$

$$= \frac{9.4}{34} \times 100$$

$$= \frac{940}{34}$$

$$= 27.65$$

$\therefore$ The coefficient of variation for the runs scored by batsman A is 27.65 (1)

Batsman B

x	$d = x - \bar{x}$	d^2
27	-8	64
35	0	0
$\boxed{35}$	0	0
37	2	4
41	6	36
175	0	104

$\bar{x} = \dfrac{175}{5} = 35$

$$\sigma = \sqrt{\frac{\sum d^2}{n}}$$

$$= \sqrt{\frac{104}{5}} = \sqrt{20.8}$$

$$\simeq 4.6$$

Coefficient of variation

$$= \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{4.6}{35} \times 100$$

$$= \frac{460}{35} = \frac{92}{7} = 13.14$$

$\therefore$ The coefficient of variation for the runs scored by batsman B is = 13.14 (2)

From (1) and (2), the coefficient of variation for B is less than the coefficient of variation for A.

$\therefore$ Batsman B is more consistent than the batsman A in scoring the runs.

Example 11.19

The mean of 30 items is 18 and their standard deviation is 3. Find the sum of all the items and also the sum of the squares of all the items.

Solution The mean of 30 items, $\bar{x} = 18$

The sum of 30 items, $\sum x = 30 \times 18 = 540$ $\left(\bar{x} = \dfrac{\sum x}{n} \right)$

Standard deviation, $\sigma = 3$

Now, $$\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$\Longrightarrow \quad \frac{\sum x^2}{30} - 18^2 = 9$$

$$\Longrightarrow \quad \frac{\sum x^2}{30} - 324 = 9$$

$$\Longrightarrow \quad \sum x^2 - 9720 = 270$$

$$\sum x^2 = 9990$$

$$\therefore \quad \sum x = 540 \text{ and } \sum x^2 = 9990.$$

Example 11.20

The mean and the standard deviation of a group of 20 items was found to be 40 and 15 respectively. While checking it was found that an item 43 was wrongly written as 53. Calculate the correct mean and standard deviation.

Solution Let us find the correct mean.

$$\text{Mean of 20 items, } \bar{x} = \frac{\sum x}{n} = 40$$

$$\Longrightarrow \quad \frac{\sum x}{20} = 40$$

$$\Longrightarrow \quad \sum x = 20 \times 40 = 800$$

$$\text{corrected } \sum x = 800 - (\text{wrong value}) + (\text{correct value})$$

$$\text{Now, } \quad \text{corrected } \sum x = 800 - 53 + 43 = 790.$$

$$\therefore \quad \text{The corrected Mean} = \frac{790}{20} = 39.5$$

$$\text{Variance, } \sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = 225 \qquad \text{(given)}$$

$$\Longrightarrow \quad \frac{\sum x^2}{20} - 40^2 = 225$$

$$\Longrightarrow \quad \sum x^2 - 32000 = 225 \times 20 = 4500$$

$$\therefore \quad \sum x^2 = 32000 + 4500 = 36500$$

$$\text{corrected } \sum x^2 = 36500 - (\text{wrong value})^2 + (\text{correct value})^2$$

$$\text{corrected } \sum x^2 = 36500 - 53^2 + 43^2 = 36500 - 2809 + 1849$$

$$= 36500 - 960 = 35540.$$

$$\text{Now, the corrected } \sigma^2 = \frac{\text{Corrected} \sum x^2}{n} - (\text{Corrected mean})^2$$

$$= \frac{35540}{20} - (39.5)^2$$

$$= 1777 - 1560.25 = 216.75$$

$$\text{Corrected } \sigma = \sqrt{216.75} \simeq 14.72$$

$$\therefore \text{ The corrected Mean} = 39.5 \text{ and the corrected S.D.} \simeq 14.72$$

Example 11.21

For a collection of data, if $\sum x = 35$, $n = 5$, $\sum (x-9)^2 = 82$, then find $\sum x^2$ and $\sum (x-\overline{x})^2$.

Solution Given that $\sum x = 35$ and $n = 5$.

$$\therefore \quad \overline{x} = \frac{\sum x}{n} = \frac{35}{5} = 7.$$

Let us find $\sum x^2$

Now, $\sum (x-9)^2 = 82$

$$\Longrightarrow \quad \sum (x^2 - 18x + 81) = 82$$

$$\Longrightarrow \quad \sum x^2 - \left(18\sum x\right) + \left(81\sum 1\right) = 82$$

$$\Longrightarrow \quad \sum x^2 - 630 + 405 = 82 \qquad \because \sum x = 35 \text{ and } \sum 1 = 5$$

$$\Longrightarrow \quad \sum x^2 = 307.$$

To find $\sum (x-\overline{x})^2$, let us consider

$$\sum (x-9)^2 = 82$$

$$\Longrightarrow \quad \sum (x-7-2)^2 = 82$$

$$\Longrightarrow \quad \sum [(x-7)-2]^2 = 82$$

$$\Longrightarrow \quad \sum (x-7)^2 - 2\sum [(x-7)\times 2] + \sum 4 = 82$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 - 4\sum (x-\overline{x}) + 4\sum 1 = 82$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 - 4(0) + (4\times 5) = 82 \qquad \because \sum 1 = 5 \text{ and } \sum (x-\overline{x}) = 0$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 = 62$$

$$\therefore \quad \sum x^2 = 307 \text{ and } \sum (x-\overline{x})^2 = 62.$$

Example 11.22

The coefficient of variations of two series are 58 and 69. Their standard deviations are 21.2 and 15.6. What are their arithmetic means?

Solution We know that coefficient of variation, $C.V = \frac{\sigma}{\overline{x}} \times 100$.

$$\therefore \quad \overline{x} = \frac{\sigma}{C.V} \times 100.$$

Mean of the first series, $\overline{x}_1 = \frac{\sigma}{C.V} \times 100$

$$= \frac{21.2}{58} \times 100 \qquad \because \quad C.V = 58 \text{ and } \sigma = 21.2$$

$$= \frac{2120}{58} = 36.6$$

Mean of the second series, $\overline{x_2} = \dfrac{\sigma}{C.V} \times 100$

$$= \frac{15.6}{69} \times 100 \qquad \because \quad C.V = 69 \text{ and } \sigma = 15.6$$

$$= \frac{1560}{69}$$

$$= 22.6$$

A.M of the I series = 36.6 and the A.M of the II series = 22.6

Exercise 11.1

1. Find the range and coefficient of range of the following data.

   (i) 59, 46, 30, 23, 27, 40, 52,35, 29

   (ii) 41.2, 33.7, 29.1, 34.5, 25.7, 24.8, 56.5, 12.5

2. The smallest value of a collection of data is 12 and the range is 59. Find the largest value of the collection of data.

3. The largest of 50 measurements is 3.84kg. If the range is 0.46kg, find the smallest measurement.

4. The standard deviation of 20 observations is $\sqrt{5}$. If each observation is multiplied by 2, find the standard deviation and variance of the resulting observations.

5. Calculate the standard deviation of the first 13 natural numbers.

6. Calculate the standard deviation of the following data.

   (i) 10, 20, 15, 8, 3, 4 (ii) 38, 40, 34 ,31, 28, 26, 34

7. Calculate the standard deviation of the following data.

x	3	8	13	18	23
f	7	10	15	10	8

8. The number of books bought at a book fair by 200 students from a school are given in the following table.

No. of books	0	1	2	3	4
No of students	35	64	68	18	15

   Calculate the standard deviation.

9. Calculate the variance of the following data

x	2	4	6	8	10	12	14	16
f	4	4	5	15	8	5	4	5

10. The time (in seconds) taken by a group of people to walk across a pedestrian crossing is given in the table below.

Time (in sec.)	5-10	10-15	15-20	20-25	25-30
No. of people	4	8	15	12	11

Calculate the variance and standard deviation of the data.

11. A group of 45 house owners contributed money towards green environment of their street. The amount of money collected is shown in the table below.

Amount (₹)	0-20	20-40	40-60	60-80	80-100
No. of house owners	2	7	12	19	5

Calculate the variance and standard deviation.

12. Find the variance of the following distribution

Class interval	20-24	25-29	30-34	35-39	40-44	45-49
Frequency	15	25	28	12	12	8

13. Mean of 100 items is 48 and their standard deviation is 10. Find the sum of all the items and the sum of the squares of all the items.

14. The mean and standard deviation of 20 items are found to be 10 and 2 respectively. At the time of checking it was found that an item 12 was wrongly entered as 8. Calculate the correct mean and standard deviation.

15. If $n = 10$, $\overline{x} = 12$ and $\sum x^2 = 1530$, then calculate the coefficient of variation .

16. Calculate the coefficient of variation of the following data: 20, 18, 32, 24, 26.

17. If the coefficient of variation of a collection of data is 57 and its S.D is 6.84, then find the mean.

18. A group of 100 candidates have their average height 163.8 cm with coefficient of variation 3.2. What is the standard deviation of their heights?

19. Given $\sum x = 99$, $n = 9$ and $\sum (x - 10)^2 = 79$. Find $\sum x^2$ and $\sum (x - \overline{x})^2$.

20. The marks scored by two students A, B in a class are given below.

A	58	51	60	65	66
B	56	87	88	46	43

Who is more consistent?

Exercise 11.2

Choose the correct answer.

1. The range of the first 10 prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23 , 29 is

   (A) 28 (B) 26 (C) 29 (D) 27

2. The least value in a collection of data is 14.1. If the range of the collection is 28.4, then the greatest value of the collection is

   (A) 42.5 (B) 43.5 (C) 42.4 (D) 42.1

3. The greatest value of a collection of data is 72 and the least value is 28. Then the coefficient of range is

   (A) 44 (B) 0.72 (C) 0.44 (D) 0.28

4 For a collection of 11 items, $\Sigma x = 132$, then the arithmetic mean is

   (A) 11 (B) 12 (C) 14 (D) 13

5. For any collection of n items, $\Sigma(x - \overline{x}) =$

   (A) Σx (B) $\overline{x}$ (C) $n\overline{x}$ (D) 0

6. For any collection of n items, $(\Sigma x) - \overline{x} =$

   (A) $n\overline{x}$ (B) $(n-2)\overline{x}$ (C) $(n-1)\overline{x}$ (D) 0

7. If t is the standard deviation of $x, y. z$, then the standard deviation of $x+5,\ y+5,\ z+5$ is

   (A) $\frac{t}{3}$ (B) $t+5$ (C) t (D) $x\,y\,z$

8. If the standard deviation of a set of data is 1.6, then the variance is

   (A) 0.4 (B) 2.56 (C) 1.96 (D) 0.04

9. If the variance of a data is 12.25, then the S.D is

   (A) 3.5 (B) 3 (C) 2.5 (D) 3.25

10. Variance of the first 11 natural numbers is

    (A) $\sqrt{5}$ (B) $\sqrt{10}$ (C) $5\sqrt{2}$ (D) 10

11. The variance of 10, 10, 10, 10, 10 is

    (A) 10 (B) $\sqrt{10}$ (C) 5 (D) 0

12. If the variance of 14, 18, 22, 26, 30 is 32, then the variance of 28, 36,44,52,60 is

    (A) 64 (B) 128 (C) $32\sqrt{2}$ (D) 32

13. Standard deviation of a collection of data is $2\sqrt{2}$. If each value is multiplied by 3, then the standard deviation of the new data is

(A) $\sqrt{12}$ (B) $4\sqrt{2}$ (C) $6\sqrt{2}$ (D) $9\sqrt{2}$

14. Given $\sum(x-\overline{x})^2 = 48$, $\overline{x} = 20$ and $n = 12$. The coefficient of variation is

(A) 25 (B) 20 (C) 30 (D) 10

15. Mean and standard deviation of a data are 48 and 12 respectively. The coefficient of variation is

(A) 42 (B) 25 (C) 28 (D) 48

Points to Remember

- (i) Range $= L - S$, the difference between the greatest and the least of the observations.

  (ii) Coefficient of range $= \dfrac{L-S}{L+S}$.

- Standard deviation for an ungrouped data

  (i) $\sigma = \sqrt{\dfrac{\sum d^2}{n}}$, where $d = x - \overline{x}$ and $\overline{x}$ is the mean.

  (ii) $\sigma = \sqrt{\dfrac{\sum d^2}{n} - \left(\dfrac{\sum d}{n}\right)^2}$, where $d = x - A$ and A is the assumed mean.

- Standard deviation for a grouped data

  (i) $\sigma = \sqrt{\dfrac{\sum fd^2}{\sum f}}$, where $d = x - \overline{x}$ and $\overline{x}$ is the mean.

  (ii) $\sigma = \sqrt{\dfrac{\sum fd^2}{\sum f} - \left(\dfrac{\sum fd}{\sum f}\right)^2}$, where $d = x - A$ and A is the assumed mean.

- Standard deviation of a collection of data remains unchanged when each value is added or subtracted by a constant.

- Standard deviation of a collection of data gets multiplied or divided by the quantity k, if each item is multiplied or divided by k.

- Standard deviation of the first n natural numbers, $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

- Variance is the square of standard deviation.

- Coefficient of variation, C.V. $= \dfrac{\sigma}{\overline{x}} \times 100$. It is used for comparing the consistency of two or more collections of data.

12

PROBABILITY

It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge
-P.D. Laplace.

- Introduction
- Classical Definition
- Addition Theorem

Pierre de Laplace
(1749-1827)
France

Laplace remembered as one of the greatest scientists of all time, sometimes referred to as a French Newton.

In 1812, Laplace established many fundamental results in statistics. He put forth a mathematical system of inductive reasoning based on probability. He only introduced the principles of probability, one among them is "probability is the ratio of the favoured events to the total possible events".

12.1 Introduction

In every day life, almost everything that we see or do is subject to chance. The occurrences of events like Earthquakes, Cyclones, Tsunami, Lightning, Epidemics, etc... are unpredictable. Most of the events occur quite unexpectedly and result in heavy loss to humanity. If we predict the occurrences of such events well in advance based on their past occurrences with a good amount of accuracy, one can think of preventive measures or damage control exercises much to the relief of human society. Such predictions well in advance of their actual happenings require the study of Probability theory.

A gambler's dispute-problem posed by Chevalier de Mere in 1654 led to exchange of letters between two famous French Mathematicians Blasie Pascal and Pierre de Fermat which created a mathematical theory of Probability. The family of major contributors to the development of Probability theory includes mathematicians like Christian Huggens (1629-1695), Bernoulli (1654-1705), De-Moivre (1667-1754), Pierre de Laplace (1749-1827), Gauss (1777-1855), Poisson (1781-1845), Chebyshev (1821-1894), Markov (1856-1922). In 1933, a Russian Mathematician A. Kolmogorov introduced an axiomatic approach which is considered as the basis for Modern Probability theory.

Probabilities always pertain to the occurrence or nonoccurrence of events. Let us define the terms random experiment, trial, sample space and different types of events used in the study of probability.

Mathematicians use the words "experiment" and "outcome" in a very wide sense. Any process of observation or measurement is called an experiment. Noting down whether a newborn baby is male or female, tossing a coin, picking up a ball from a bag containing balls of different colours and observing the number of accidents at a particular place in a day are some examples of experiments.

A random experiment is one in which the exact outcome cannot be predicted before conducting the experiment. However, one can list out all possible outcomes of the experiment.

The set of all possible outcomes of a random experiment is called its sample space and it is denoted by the letter S. Each repetition of the experiment is called a trial.

A subset of the sample space S is called an event.

Let A be a subset of S. If the experiment, when conducted, results in an outcome that belongs to A, then we say that the event A has occurred.

Let us illustrate random experiment, sample space, events with the help of some examples.

Random Experiment	Sample Space	Some Events
Tossing an unbiased coin once	$S = \{H, T\}$	The occurrence of head, $\{H\}$ is an event. The occurrence of tail, $\{T\}$ is another event.
Tossing an unbiased coin twice	$S = \{HT, HH, TT, TH\}$	$\{HT, HH\}$ and $\{TT\}$ are some of the events
Rolling an unbiased die once	$S = \{1, 2, 3, 4, 5, 6\}$	$\{1,3,5\}, \{2,4,6\}, \{3\}$ and $\{6\}$ are some of the events

Equally likely events

Two or more events are said to be equally likely if each one of them has an equal chance of occurrence.

In tossing a coin, the occurrence of Head and the occurrence of Tail are equally likely events.

Mutually exclusive events

Two or more events are said to be mutually exclusive if the occurrence of one event prevents the occurrence of other events. That is, mutually exclusive events can't occur simultaneously. Thus, if A and B are two mutually exclusive events, then $A \cap B = \phi$.

Fig. 12.1

In tossing a coin, the occurrence of head excludes the occurrence of tail. Similarly if an unbiased die is rolled, the six possible outcomes are mutually exclusive, since two or more faces cannot turn up simultaneously.

Complementary events

Let E be an event of a random experiment and S be its sample space. The set containing all the other outcomes which are not in E but in the sample space is called the complimentary event of E. It is denoted by $\overline{E}$. Thus, $\overline{E} = S - E$. Note that E and $\overline{E}$ are mutually exclusive events.

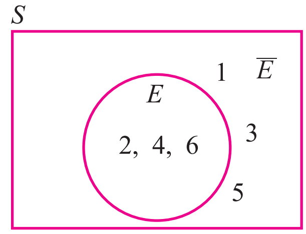


Fig. 12.2

In throwing a die, let $E = \{2, 4, 6\}$ be an event of getting a multiple of 2.

Then the complementary of the event E is given by $\overline{E} = \{1, 3, 5\}$.(see Figure 12.2)

Exhaustive events

Events $E_1, E_2, \cdots, E_n$ are exhaustive events if their union is the sample space S.

Sure event

The sample space of a random experiment is called sure or certain event as any one of its elements will surely occur in any trail of the experiment.

For example, getting one of 1, 2, 3, 4, 5, 6 in rolling a die is a sure event.

Impossible event

An event which will not occur on any account is called an impossible event.

It is denoted by ϕ.

For example, getting 7 in rolling a die once is an impossible event.

Favourable outcomes

The outcomes corresponding to the occurrence of the desired event are called favourable outcomes of the event.

For example, if E is an event of getting an odd number in rolling a die, then the outcomes 1, 3, 5 are favourable to the event E.

Note

In this chapter, we consider only random experiments all of whose outcomes are equally likely and sample spaces are finite. Thus, whenever we refer coins or dice, they are assumed to be unbiased.

12.2 Classical definition of probability

If a sample space contains n outcomes and if m of them are favourable to an event A, then, we write $n(S) = n$ and $n(A) = m$. The Probability of the event A, denoted by $P(A)$, is defined as the ratio of m to n.

That is $P(A) = \dfrac{\text{number of outcomes favourable to } A}{\text{total number of outcomes}}$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}.$$

Note

(i) The above classical definition of probability is not applicable if the number of possible outcomes is infinite and the outcomes are not equally likely.

(ii) The probability of an event A lies between 0 and 1, both inclusive;

That is $0 \leq P(A) \leq 1$.

(iii) The probability of the sure event is 1. That is $P(S) = 1$.

(iv) The probability of an impossible event is 0. That is $P(\phi) = 0$.

(v) The probability that the event A will not occur is given by

$$P(\text{not } A) = P(\overline{A}) \text{ or } P(A') = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n}$$

$$\Longrightarrow \quad P(\overline{A}) = 1 - \frac{m}{n} = 1 - P(A).$$

(vi) $P(A) + P(\overline{A}) = 1$.

Example 12.1

A fair die is rolled. Find the probability of getting

(i) the number 4 (ii) an even number

(iii) a prime factor of 6 (iv) a number greater than 4.

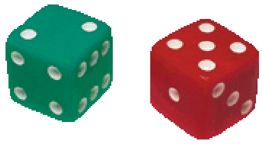


Fig. 12.3

Solution In rolling a die, the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore n(S) = 6$.

(i) Let A be the event of getting 4.

$A = \{4\}$ $\therefore n(A) = 1$.

$\therefore P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{6}$.

(ii) Let B be the event of getting an even number.

$B = \{2, 4, 6\}$ $\therefore$ $n(B) = 3$.

Hence $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$.

(iii) Let C be the event of getting a prime factor of 6.

Then $C = \{2, 3\}$ $\therefore$ $n(C) = 2$.

Hence $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

(iv) Let D be the event of getting a number greater than 4.

$D = \{5, 6\}$ $n(D) = 2$.

Hence, $P(D) = \dfrac{n(D)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

Example 12.2

In tossing a fair coin twice, find the probability of getting

(i) two heads (ii) atleast one head (iii) exactly one tail

Solution In tossing a coin twice, the sample space

$$S = \{HH, HT, TH, TT\}$$

$\therefore$ $n(S) = 4$.

(i) Let A be the event of getting two heads. Then $A = \{\,HH\,\}$.

Thus, $n(A) = 1$.

$\therefore$ $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{4}$.

(ii) Let B be the event of getting at least one head. Then $B = \{\,HH, HT, TH\,\}$

Thus, $n(B) = 3$.

$\therefore$ $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{4}$.

(iii) Let C be the event of getting exactly one tail. Then $C = \{\,HT, TH\,\}$

Thus, $n(C) = 2$.

$\therefore$ $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$.

Example 12.3

An integer is chosen from the first twenty natural numbers. What is the probability that it is a prime number?

Solution Here $S = \{\,1, 2, 3, \cdots, 20\,\}$

$\therefore$ $n(S) = 20$.

Let A be the event of choosing a prime number.

Then, $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

$n(A) = 8$.

Hence, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{8}{20} = \dfrac{2}{5}$.

Example 12.4

There are 7 defective items in a sample of 35 items. Find the probability that an item chosen at random is non-defective.

Solution Total number of items $n(S) = 35$.

Number of defective items = 7.

Let A be the event of choosing a non-defective item.

Number of non-defective items, $n(A) = 35 - 7 = 28$.

$\therefore$ Probability that the chosen item is non-defective,

$$P(A) = \frac{n(A)}{n(S)} = \frac{28}{35} = \frac{4}{5}.$$

Example 12.5

Two unbiased dice are rolled once. Find the probability of getting

(i) a sum 8 (ii) a doublet (iii) a sum greater than 8.

Solution When two dice are thrown, the sample space is

$$\begin{aligned} S = \{ &(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),\\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\ &(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),\\ &(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \} \end{aligned}$$

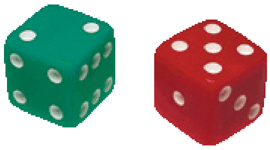


Fig. 12.4

$\therefore n(S) = 6 \times 6 = 36$.

(i) Let A be the event of getting a sum 8.

$\therefore \quad A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.

Then $\quad n(A) = 5$.

Hence, $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{36}$.

(ii) Let B be the event of getting a doublet

$\therefore \quad B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$.

Thus, $\quad n(B) = 6$.

$\therefore \quad P(B) = \dfrac{n(B)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$.

(iii) Let C be the event of getting a sum greater than 8.

Then, $\quad C = \{(3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6)\}$.

Thus, $\quad n(C) = 10$.

$\therefore \quad P(C) = \dfrac{n(C)}{n(S)} = \dfrac{10}{36} = \dfrac{5}{18}$.

Example 12.6

From a well shuffled pack of 52 playing cards, one card is drawn at random. Find the probability of getting

(i) a king (ii) a black king

(iii) a spade card (iv) a diamond 10.

The 52 playing cards are classified as

Spade ♠	Hearts ♥	Clavor ♣	Diamond ♦
A	A	A	A
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9
10	10	10	10
J	J	J	J
Q	Q	Q	Q
K	K	K	K
13	13	13	13

Solution Now, $n(S) = 52$.

(i) Let A be the event of drawing a king card

$$\therefore n(A) = 4.$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

(ii) Let B be the event of drawing a black king card

Thus, $n(B) = 2$.

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{2}{52} = \frac{1}{26}.$$

(iii) Let C be the event of drawing a spade card

Thus, $n(C) = 13$.

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{13}{52} = \frac{1}{4}.$$

(iv) Let D be the event of drawing a diamond 10 card.

Thus, $n(D) = 1$.

$$P(D) = \frac{n(D)}{n(S)} = \frac{1}{52}.$$

Example 12.7

There are 20 boys and 15 girls in a class of 35 students . A student is chosen at random. Find the probability that the chosen student is a (i) boy (ii) girl.

Solution Let S be the sample space of the experiment.

Let B and G be the events of selecting a boy and a girl respectively.

$$\therefore \; n(S) = 35, \; n(B) = 20 \text{ and } n(G) = 15.$$

(i) Probability of choosing a boy is $P(B) = \frac{n(B)}{n(S)} = \frac{20}{35}$

$$\Longrightarrow P(B) = \frac{4}{7}.$$

(ii) Probability of choosing a girl is $P(G) = \frac{n(G)}{n(S)} = \frac{15}{35}$

$$\Longrightarrow P(G) = \frac{3}{7}.$$

Example 12.8

The probability that it will rain on a particular day is 0.76. What is the probability that it will not rain on that day?

Solution Let A be the event that it will rain. Then $\overline{A}$ is the event that it will not rain.

Given that $P(A) = 0.76.$

Thus, $P(\overline{A}) = 1 - 0.76 \qquad \because P(A) + P(\overline{A}) = 1$

$= 0.24.$

$\therefore$ The probability that it will not rain is 0.24.

Example 12.9

A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball from the bag is thrice that of drawing a red ball, then find the number of blue balls in the bag.

Solution Let the number of blue balls be x.

$\therefore$ Total number of balls, $n(S) = 5 + x$.

Let B be the event of drawing a blue ball and R be the event of drawing a red ball.

$$\text{Given} \qquad P(B) = 3P(R)$$

$$\implies \qquad \frac{n(B)}{n(S)} = 3\frac{n(R)}{n(S)}$$

$$\implies \qquad \frac{x}{5+x} = 3\left(\frac{5}{5+x}\right)$$

$$\implies \qquad x = 15$$

Thus, number of blue balls $= 15$.

Example 12.10

Find the probability that

(i) a leap year selected at random will have 53 Fridays

(ii) a leap year selected at random will have only 52 Fridays

(iii) a non-leap year selected at random will have 53 Fridays.

Solution (i) Number of days in a leap year = 366 days. i.e., 52 weeks and 2 days.

Now 52 weeks contain 52 Fridays and the remaining two days will be one of the following seven possibilities.

(Sun, Mon), (Mon, Tue), (Tue, Wed), (Wed, Thur), (Thur, Fri), (Fri, Sat) and (Sat, Sun).

The probability of getting 53 Fridays in a leap year is same as the probability of getting a Friday in the above seven possibilities.

Here $S = \{(\text{Sun},\text{Mon}),(\text{Mon},\text{Tue}),(\text{Tue},\text{Wed}),(\text{Wed},\text{Thur}),(\text{Thur},\text{Fri}),(\text{Fri},\text{Sat}),(\text{Sat},\text{Sun})\}$.

Then $n(S) = 7$.

Let A be the event of getting one Friday in the remaining two days.

$A = \{(Thur, Fri),(Fri, Sat)\}$ Then $n(A) = 2$.

$$p(A) = \frac{n(A)}{n(S)} = \frac{2}{7}.$$

(ii) To get only 52 Fridays in a leap year, there must be no Friday in the remaining two days. Let B be the event of not getting a Friday in the remaining two days. Then

$$B = \{(\text{Sun},\text{Mon}),(\text{Mon},\text{Tue}),(\text{Tue},\text{Wed}),(\text{Wed},\text{Thur}),(\text{Sat},\text{Sun})\}.$$

$$n(B) = 5.$$

Now, $P(B) = \frac{n(B)}{n(S)} = \frac{5}{7}$.

Note that A and B are complementary events.

(iii) Number of days in a non leap year = 365 days. i.e., 52 weeks and 1 day.

To get 53 Fridays in a non leap year, there must be a Friday in the seven possibilities: Sun, Mon, Tue, Wed, Thur, Fri and Sat.

Here $\text{S} = \{\text{Sun, Mon, Tue, Wed, Thur, Fri and Sat}\}$.

$\therefore$ $n(S) = 7$.

Let C be the event of getting a Friday in the remaining one day . Then

$$C = \{\text{Fri}\} \implies n(C) = 1.$$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{1}{7}.$$

Example 12.11

If A is an event of a random experiment such that $P(A) : P(\overline{A}) = 7:12$, then find $P(A)$.

Solution Given that $P(A) : P(\overline{A}) = 7:12$.

Let $P(A) = 7k$ and $P(\overline{A}) = 12k$., $k > 0$

We know that $P(A) + P(\overline{A}) = 1$.

Then, $7k + 12k = 1 \implies 19k = 1$.

Thus, $k = \frac{1}{19}$

$\therefore$ $P(A) = 7k = \frac{7}{19}$.

Aliter

$$\frac{P(A)}{P(\overline{A})} = \frac{7}{12}$$

$$12P(A) = 7 \times P(\overline{A})$$

$$= 7\,[1 - P(A)]$$

$$19P(A) = 7$$

Thus, $P(A) = \frac{7}{19}$

Exercise 12. 1

1. A ticket is drawn from a bag containing 100 tickets. The tickets are numbered from one to hundred. What is the probability of getting a ticket with a number divisible by 10?
2. A die is thrown twice. Find the probability of getting a total of 9.
3. Two dice are thrown together. Find the probability that the two digit number formed with the two numbers turning up is divisible by 3.
4. Three rotten eggs are mixed with 12 good ones. One egg is chosen at random. What is the probability of choosing a rotten egg?
5. Two coins are tossed together. What is the probability of getting at most one head.
6. One card is drawn randomly from a well shuffled deck of 52 playing cards. Find the probability that the drawn card is

   (i) a Diamond (ii) not a Diamond (iii) not an Ace.
7. Three coins are tossed simultaneously. Find the probability of getting

   (i) at least one head (ii) exactly two tails (iii) at least two heads.
8. A bag contains 6 white balls numbered from 1 to 6 and 4 red balls numbered from 7 to 10. A ball is drawn at random. Find the probability of getting

   (i) an even-numbered ball (ii) a white ball.
9. A number is selected at random from integers 1 to 100. Find the probability that it is

   (i) a perfect square (ii) not a perfect cube.
10. For a sightseeing trip, a tourist selects a country randomly from Argentina, Bangladesh, China, Angola, Russia and Algeria. What is the probability that the name of the selected country will begin with A ?
11. A box contains 4 Green, 5 Blue and 3 Red balls. A ball is drawn at random. Find the probability that the selected ball is (i) Red in colour (ii) not Green in colour.
12. 20 cards are numbered from 1 to 20. One card is drawn at random. What is the probability that the number on the card is

    (i) a multiple of 4 (ii) not a multiple of 6.
13. A two digit number is formed with the digits 3, 5 and 7. Find the probability that the number so formed is greater than 57 (repetition of digits is not allowed).
14. Three dice are thrown simultaneously. Find the probability of getting the same number on all the three dice.

15. Two dice are rolled and the product of the outcomes (numbers) are found. What is the probability that the product so found is a prime number?

16. A jar contains 54 marbles each of which is in one of the colours blue, green and white. The probability of drawing a blue marble is $\frac{1}{3}$ and the probability of drawing a green marble is $\frac{4}{9}$. How many white marbles does the jar contain?

17. A bag consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. A trader A will accept only the shirt which are good, but the trader B will not accept the shirts which have major defects. One shirt is drawn at random. What is the probability that it is acceptable by (i) A (ii) B ?

18. A bag contains 12 balls out of which x balls are white. (i) If one ball is drawn at random, what is the probability that it will be a white ball. (ii) If 6 more white balls are put in the bag and if the probability of drawing a white ball will be twice that of in (i), then find x.

19. Piggy bank contains 100 fifty-paise coins, 50 one-rupee coins, 20 two-rupees coins and 10 five- rupees coins. One coin is drawn at random. Find the probability that the drawn coin (i) will be a fifty-paise coin (ii) will not be a five-rupees coin.

12.3 Addition theorem on probability

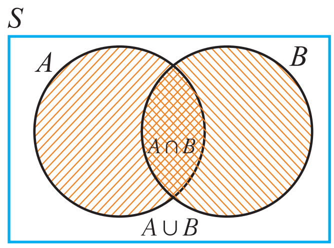


Fig. 12.5

Let A and B be subsets of a finite non-empty set S. Then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Divide both sides by $n(S)$,we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \qquad (1)$$

If the subsets A and B correspond to two events A and B of a random experiment and if the set S corresponds to the sample space S of the experiment, then (1) becomes

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This result is known as the addition theorem on probability.

Note

(i) The event $A \cup B$ is said to occur if the event A occurs or the event B occurs or both A and B occur simultaneously. The event $A \cap B$ is said to occur if both the events A and B occur simultaneously.

(ii) If A and B are mutually exclusive events, then $A \cap B = \emptyset$.

Thus, $P(A \cup B) = P(A) + P(B) \qquad \because P(A \cap B) = 0$.

(iii) $A \cap \overline{B}$ is same as $A \setminus B$ in the language of set theory.

Results (without proof)

(i) If A, B and C are any 3 events associated with a sample space S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

(ii) If A_1, A_2 and A_3 are three mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3).$$

(iii) If $A_1, A_2, A_3, \cdots, A_n$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n).$$

(iv) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$,

$P(\overline{A} \cap B) = P(B) - P(A \cap B)$

where $A \cap \overline{B}$ mean only A and not B;

Similarly $\overline{A} \cap B$ means only B and not A.

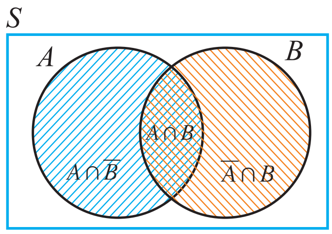


Fig. 12.6

Example 12.12

Three coins are tossed simultaneously. Using addition theorem on probability, find the probability that either exactly two tails or at least one head turn up.

Solution Now the sample space $S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$.

Hence, $n(S) = 8$.

Let A be the event of getting exactly two tails.

Thus, $A = \{HTT, TTH, THT\}$ and hence $n(A) = 3$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}.$$

Let B be the event of getting at least one head.

Thus, $B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$ and hence $n(B) = 7$.

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}.$$

Now, the events A and B are not mutually exclusive.

Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{3}{8}$.

$\therefore$ $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

Thus $P(A \cup B) = \frac{3}{8} + \frac{7}{8} - \frac{3}{8} = \frac{7}{8}$.

Note

In the above problem, we applied addition theorem on probability.

However, one can notice that $A \cup B = B$. Thus, $P(A \cup B) = P(B) = \frac{7}{8}$.

Example 12.13

A die is thrown twice. Find the probability that at least one of the two throws comes up with the number 5 (use addition theorem).

Solution In rolling a die twice, the size of the sample space , $n(S) = 36$.

Let A be the event of getting 5 in the first throw.

$\therefore \quad A = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$.

Thus, $\quad n(A) = 6$, and $\quad P(A) = \frac{6}{36}$.

Let B be the event of getting 5 in the second throw.

$\therefore \quad B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5)\}$.

Thus, $\quad n(B) = 6$ and $\quad P(B) = \frac{6}{36}$.

A and B are not mutually exclusive events , since $\quad A \cap B = \{(5,5)\}$.

$\therefore \quad n(A \cap B) = 1$ and $\quad P(A \cap B) = \frac{1}{36}$.

$\therefore$ By addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}.$$

Example 12.14

The probability that a girl will be selected for admission in a medical college is 0.16. The probability that she will be selected for admission in an engineering college is 0.24 and the probability that she will be selected in both, is 0.11

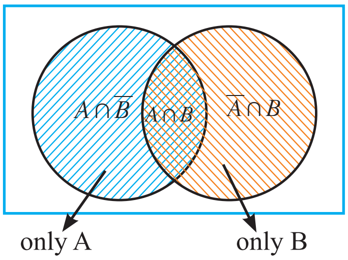


Fig. 12.7

(i) Find the probability that she will be selected in at least one of the two colleges.

(ii) Find the probability that she will be selected either in a medical college only or in an engineering college only.

Solution Let A be the event of getting selected in a medical college and B be the event of getting selected for admission in an engineering college.

(i) $\quad P(A) = 0.16$, $P(B) = 0.24$ and $P(A \cap B) = 0.11$

P (she will be selected for admission in at least one of the two colleges) is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.16 + 0.24 - 0.11 = 0.29$$

(ii) P(she will be selected for admission in only one of the two colleges)

$$= P\text{ (only } A \text{ or only } B)$$
$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$$
$$= (0.16 - 0.11) + (0.24 - 0.11) = 0.18.$$

Example 12.15

A letter is chosen at random from the letters of the word "ENTERTAINMENT". Find the probability that the chosen letter is a vowel or T. (repetition of letters is allowed)

Solution There are 13 letters in the word ENTERTAINMENT.

$\therefore \quad n(S) = 13.$

Let A be the event of getting a vowel.

$\therefore \quad n(A) = 5.$

Hence, $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{13}.$

Let B be the event of getting the letter T.

$\therefore \quad n(B) = 3$

Hence, $\quad P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{13}$. Then

$P(A \text{ or } B) = P(A) + P(B) \qquad \because A$ and B are mutually exclusive events

$$= \frac{5}{13} + \frac{3}{13} = \frac{8}{13}.$$

Example 12.16

Let A, B, C be any three mutually exclusive and exhaustive events such that $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$. Find $P(A)$.

Solution

Let $\quad P(A) = p.$

Now, $P(B) = \frac{3}{2}P(A) = \frac{3}{2}p.$

Also, $P(C) = \frac{1}{2}P(B) = \frac{1}{2}\left(\frac{3}{2}p\right) = \frac{3}{4}p.$

Given that A, B and C are mutually exclusive and exhaustive events.

$\therefore\ P(A \cup B \cup C) = P(A) + P(B) + P(C)$ and $S = A \cup B \cup C$.

Now, $\quad P(S) = 1.$

That is, $\quad P(A)+P(B)+P(C)=1$

$\Longrightarrow \quad p+\frac{3}{2}p+\frac{3}{4}p=1$

$\Longrightarrow \quad 4p+6p+3p=4$

Thus, $\quad p=\frac{4}{13}$

Hence, $\quad P(A)=\frac{4}{13}.$

Example 12.17

A card is drawn from a deck of 52 cards. Find the probability of getting a King or a Heart or a Red card.

Solution Let A, B and C be the events of getting a King, a Heart and a Red card respectively.

Now, $n(S)=52$, $n(A)=4$, $n(B)=13$, $n(C)=26$. Also,

$n(A\cap B)=1$, $n(B\cap C)=13$, $n(C\cap A)=2$ and $n(A\cap B\cap C)=1$.

$\therefore\ P(A)=\frac{4}{52}$, $P(B)=\frac{13}{52}$, $P(C)=\frac{26}{52}$.

$P(A\cap B)=\frac{1}{52}$, $P(B\cap C)=\frac{13}{52}$, $P(C\cap A)=\frac{2}{52}$ and $P(A\cap B\cap C)=\frac{1}{52}$.

Now $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

$$=\frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52}=\frac{44-16}{52}$$

$$=\frac{7}{13}.$$

Example 12.18

A bag contains 10 white, 5 black, 3 green and 2 red balls. One ball is drawn at random. Find the probability that the ball drawn is white or black or green.

Solution Let S be the sample space.

$\therefore\quad n(S)=20.$

Let W, B and G be the events of selecting a white, black and green ball respectively.

Probability of getting a white ball, $P(W)=\frac{n(W)}{n(S)}=\frac{10}{20}.$

Probability of getting a black ball, $P(B)=\frac{n(B)}{n(S)}=\frac{5}{20}.$

Probability of getting a green ball, $P(G)=\frac{n(G)}{n(S)}=\frac{3}{20}.$

$\therefore$ Probability of getting a white or black or green ball,

$P(W\cup B\cup G)=P(W)+P(B)+P(G)$ $\quad\because$ W, B and G are mutually exclusive.

$$=\frac{10}{20}+\frac{5}{20}+\frac{3}{20}=\frac{9}{10}.$$

(Note : $P(W\cup B\cup G)=P(R')=1-P(R)=1-\frac{2}{20}=\frac{9}{10}$)

Exercise 12.2

1. If A and B are mutually exclusive events such that $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, then find $P(A \cup B)$.
2. If A and B are two events such that $P(A) = \frac{1}{4}, P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$,then find $P(A \cap B)$.
3. If $P(A) = \frac{1}{2}, P(B) = \frac{7}{10}, P(A \cup B) = 1$. Find (i) $P(A \cap B)$ (ii) $P(A' \cup B')$.
4. If a die is rolled twice, find the probability of getting an even number in the first time or a total of 8 .
5. One number is chosen randomly from the integers 1 to 50. Find the probability that it is divisible by 4 or 6.
6. A bag contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If an item is chosen at random, find the probability that it is rusted or that it is a bolt.
7. Two dice are rolled simultaneously. Find the probability that the sum of the numbers on the faces is neither divisible by 3 nor by 4.
8. A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are rotten. If a person takes out one fruit at random, find the probability that the fruit is either an apple or a good fruit.
9. In a class, 40% of the students participated in Mathematics-quiz, 30% in Science-quiz and 10% in both the quiz programmes. If a student is selected at random from the class, find the probability that the student participated in Mathematics or Science or both quiz programmes.
10. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that it will be a spade or a king.
11. A box contains 10 white, 6 red and 10 black balls. A ball is drawn at random. Find the probability that the ball drawn is white or red.
12. A two digit number is formed with the digits 2, 5, 9 (repetition is allowed). Find the probability that the number is divisible by 2 or 5.
13. Each individual letter of the word "ACCOMMODATION" is written in a piece of paper, and all 13 pieces of papers are placed in a jar. If one piece of paper is selected at random from the jar, find the probability that

    (i) the letter 'A' or 'O' is selected.

    (ii) the letter 'M' or 'C' is selected.

14. The probability that a new car will get an award for its design is 0.25, the probability that it will get an award for efficient use of fuel is 0.35 and the probability that it will get both the awards is 0.15. Find the probability that

(i) it will get atleast one of the two awards

(ii) it will get only one of the awards.

15. The probability that A, B and C can solve a problem are $\frac{4}{5}, \frac{2}{3}$ and $\frac{3}{7}$ respectively. The probability of the problem being solved by A and B is $\frac{8}{15}$, B and C is $\frac{2}{7}$, A and C is $\frac{12}{35}$. The probability of the problem being solved by all the three is $\frac{8}{35}$. Find the probability that the problem can be solved by atleast one of them.

Exercise 12.3

Choose the correct answer

1. If ϕ is an impossible event, then $P(\phi) =$

(A) 1 (B) $\frac{1}{4}$ (C) 0 (D) $\frac{1}{2}$

2. If S is the sample space of a random experiment, then $P(S) =$

(A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) 1

3. If p is the probability of an event A, then p satisfies

(A) $0 < p < 1$ (B) $0 \le p \le 1$ (C) $0 \le p < 1$ (D) $0 < p \le 1$

4. Let A and B be any two events and S be the corresponding sample space. Then $P(\overline{A} \cap B) =$

(A) $P(B) - P(A \cap B)$ (B) $P(A \cap B) - P(B)$

(C) $P(S)$ (D) $P[(A \cup B)']$

5. The probability that a student will score centum in mathematics is $\frac{4}{5}$. The probability that he will not score centum is

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$

6. If A and B are two events such that $P(A) = 0.25, P(B) = 0.05$ and $P(A \cap B) = 0.14$, then $P(A \cup B) =$

(A) 0.61 (B) 0.16 (C) 0.14 (D) 0.6

7. There are 6 defective items in a sample of 20 items. One item is drawn at random. The probability that it is a non-defective item is

(A) $\frac{7}{10}$ (B) 0 (C) $\frac{3}{10}$ (D) $\frac{2}{3}$

8. If A and B are mutually exclusive events and S is the sample space such that $P(A) = \frac{1}{3}P(B)$ and $S = A \cup B$, then $P(A) =$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{8}$

9. The probabilities of three mutually exclusive events A, B and C are given by $\frac{1}{3}, \frac{1}{4}$, and $\frac{5}{12}$. Then $P(A \cup B \cup C)$ is

(A) $\frac{19}{12}$ (B) $\frac{11}{12}$ (C) $\frac{7}{12}$ (D) 1

10. If $P(A) = 0.25, P(B) = 0.50, P(A \cap B) = 0.14$ then $P(\text{neither } A \text{ nor } B) =$

(A) 0.39 (B) 0.25 (C) 0.11 (D) 0.24

11. A bag contains 5 black balls, 4 white balls and 3 red balls. If a ball is selected at random, the probability that it is not red is

(A) $\frac{5}{12}$ (B) $\frac{4}{12}$ (C) $\frac{3}{12}$ (D) $\frac{3}{4}$

12. Two dice are thrown simultaneously. The probability of getting a doublet is

(A) $\frac{1}{36}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{2}{3}$

13. A fair die is thrown once. The probability of getting a prime or composite number is

(A) 1 (B) 0 (C) $\frac{5}{6}$ (D) $\frac{1}{6}$

14. Probability of getting 3 heads or 3 tails in tossing a coin 3 times is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2}$

15. A card is drawn from a pack of 52 cards at random. The probability of getting neither an ace nor a king card is

(A) $\frac{2}{13}$ (B) $\frac{11}{13}$ (C) $\frac{4}{13}$ (D) $\frac{8}{13}$

16. The probability that a leap year will have 53 Fridays or 53 Saturdays is

(A) $\frac{2}{7}$ (B) $\frac{1}{7}$ (C) $\frac{4}{7}$ (D) $\frac{3}{7}$

17. The probability that a non-leap year will have 53 Sundays and 53 Mondays is

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{3}{7}$ (D) 0

18. The probability of selecting a queen of hearts when a card is drawn from a pack of 52 playing cards is

(A) $\frac{1}{52}$ (B) $\frac{16}{52}$ (C) $\frac{1}{13}$ (D) $\frac{1}{26}$

19. Probability of sure event is

(A) 1 (B) 0 (C) 100 (D) 0.1

20. The outcome of a random experiment results in either success or failure. If the probability of success is twice the probability of failure, then the probability of success is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) 0

Answers

1. SETS AND FUNCTIONS

Exercise 1.1

2. (i) A (ii) ϕ **3.** (i) {b, c} (ii) ϕ (iii) $\{a, e, f, s\}$

4. (i) {2, 4, 6, 7, 8, 9} (ii) {4, 6} (iii) {4, 6, 7, 8, 9}

10. {–5, –3, –2}, {–5, –3}, not associative

Exercise 1.2

2. Different answers are possible for (i) to (iv). One such answer is :

(i) $A' \cup (A \cap B)$ or $(A \backslash B)'$ (ii) $(A \cap B) \cup (A \cap C)$ (iii) $A \backslash (B \cup C)$ (iv) $(A \cap B) \backslash C$

5. (i) {12} (ii) {4, 8, 12, 20, 24, 28}

Exercise 1.3

1. 300 **2.** 430 **3.** 35 **5.** 100 **6.** 10%

7. (i) 10 (ii) 25 (iii) 15 **8.** (i) 450 (ii) 3550 (iii) 1850 **9.** 15

Exercise 1.4

1. (i) not a function (ii) function **2.** domain = {1, 2, 3, 4, 5}; range = {1, 3, 5, 7, 9}

3. (i) neither one to one nor onto (ii) constant function (iii) one-one and onto function

4. (i) not a function (ii) one-one function (iii) not a function (iv) bijective

5. $a = -2,\ b = -5,\ c = 8,\ d = -1$ **6.** range is $\left\{-\frac{1}{2}, -1, 1, \frac{1}{2}\right\}$; f is not a function from A to A

7. one-one and onto function **8.** (i) 12 and 14 (ii) 13 and 15 **9.** $a = 9,\ b = 15$

10. (i) $f = \{(5, -7), (6, -9), (7, -11), (8, -13)\}$

(ii) co-domain $= \{-11, 4, 7, -10, -7, -9, -13\}$

(iii) range $= \{-7, -9, -11, -13\}$ (iv) one-one function

11. (i) function (ii) function (iii) not a function (iv) not a function (v) function

12.

x	–1	–3	–5	–4
$f(x)$	2	1	6	3

13. $\{(6,1), (9,2), (15,4), (18,5), (21,6)\}$

x	6	9	15	18	21
$f(x)$	1	2	4	5	6

14. $\{(4,3),(6,4),(8,5),(10,6)\}$

x	4	6	8	10
$f(x)$	3	4	5	6

15. (i) 16 (ii) –32 (iii) 5 (iv) $\frac{2}{3}$ **16.** (i) 23 (ii) 34 (iii) 2

Exercise 1.5

1	2	3	4	5	6	7	8	9	10
A	C	C	A	A	B	A	B	B	B
11	12	13	14	15	16	17	18	19	20
A	B	C	D	A	D	D	B	A	C

2. SEQUENCES AND SERIES OF REAL NUMBERS

Exercise 2.1

1. (i) $-\frac{1}{3}, 0, 1$ (ii) $-27,\ 81,\ -243$ (iii) $-\frac{3}{4},\ 2,\ -\frac{15}{4}$

2. (i) $\frac{9}{17}, \frac{11}{21}$ (ii) –1536, 18432 (iii) 36, 78 (iv) –21, 57

3. 378, $\frac{25}{313}$ **4.** 195, 256 **5.** 2, 5, 15, 35, 75 **6.** 1, 1, 1, 2, 3, 5

Exercise 2.2

1. A.P : 6, 11, 16, $\cdots$; the general term is $5n+1$ **2.** common difference is –5, $t_{15} = 55$

3. $t_{29} = 3$ **4.** $t_{12} = 23\sqrt{2}$ **5.** $t_{17} = 84$ **6.** (i) 27 terms (ii) 34 terms

8. $t_{27} = 109$ **9.** $n = 10$ **10.** 7 **11.** First year : 100, $t_{15} = 2200$

12. 2560 **13.** 10, 2, –6 or –6, 2, 10 **14.** 2, 6, 10 or 10, 6, 2 **16.** A.P., ₹95,000

Exercise 2.3

1. (i) G.P. with $r = 2$ (ii) G.P. with $r = 5$ (iii) G.P. with $r = \frac{2}{3}$

(iv) G.P. with $r = \frac{1}{12}$ (v) G.P. with $r = \frac{1}{2}$ (vi) not a G.P.

2. -2^7 **3.** 2, 6, 18 , $\cdots$ **4.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$ **5.** (i) $n = 8$ (ii) $n = 11$ **6.** $n = 5$

7. $r = 5$ **8.** $r = \frac{5}{2}$ or $\frac{2}{5}$; the terms : $\frac{2}{5}, 1, \frac{5}{2}$. (or) $\frac{5}{2}, 1, \frac{2}{5}$. **9.** 18, 6, 2 (or) 2, 6, 18

10. 4, 2, 1 (or) 1, 2, 4 **11.** $1, 3, 9, \cdots$ (or) $9, 3, 1, \cdots$ **12.** ₹$1000\left(\frac{105}{100}\right)^{12}$ **13.** ₹$50,000\left(\frac{85}{100}\right)^{15}$

Exercise 2.4

1. (i) 2850 (ii) 7875 **2.** 1020 **3.** (i) 260 (ii) -75 **4.** (i) 1890 (ii) 50 5. -820

6. $\frac{39}{11} + \frac{40}{11} + \frac{41}{11} + \cdots$ **7.** 8 terms or 23 terms **8.** 55350 **9.** 740 **10.** 7227 **11.** 36

12. 12000 **13.** 15 days **14.** A.P., ₹37,200 **16.** 156 times **20.** 1225 bricks

Exercise 2.5

1. $s_{20} = \frac{15}{4}\left[1-\left(\frac{1}{3}\right)^{20}\right]$ **2.** $s_{27} = \frac{1}{6}\left[1-\left(\frac{1}{3}\right)^{27}\right]$ **3.** (i) 765 (ii) $\frac{5}{2}(3^{12}-1)$

4. (i) $\frac{1-(0.1)^{10}}{0.9}$ (ii) $\frac{10}{81}(10^{20}-1)-\frac{20}{9}$ **5.** (i) $n=6$ (ii) $n=6$ **6.** $\frac{75}{4}\left[1-\left(\frac{4}{5}\right)^{23}\right]$

7. $3+6+12+\cdots$ **8.** (i) $\frac{70}{81}[10^n-1]-\frac{7n}{9}$ (ii) $n-\frac{2}{3}\left[1-\left(\frac{1}{10}\right)^n\right]$

9. $s_{15} = \frac{5(4^{15}-1)}{3}$ **10.** 2^{nd} option; number of mangoes 1023. **11.** $r=2$

Exercise 2.6

1. (i) 1035 (ii) 4285 (iii) 2550 (iv) 17395 (v) 10650 (vi) 382500

2. (i) $k=12$ (ii) $k=9$ **3.** 29241 **4.** 91 **5.** 3818 cm^2 **6.** 201825 cm^3

Exercise 2.7

1	2	3	4	5	6	7	8	9	10
B	D	C	D	D	A	B	B	B	B
11	12	13	14	15	16	17	18	19	20
B	A	B	D	A	B	B	A	C	A

3. ALGEBRA

Exercise 3.1

1. $\left(4,\frac{3}{2}\right)$ **2.** $(1,5)$ **3.** $(3,2)$ **4.** $\left(\frac{1}{3},\frac{1}{2}\right)$ **5.** $(1,5)$

6. $\left(\frac{11}{23},\frac{22}{31}\right)$ **7.** $(2,4)$ **8.** $(2,1)$ **9.** $\left(5,\frac{1}{7}\right)$ **10.** $(6,-4)$

Exercise 3.2

1. (i) $(4,3)$ (ii) $(0.4, 0.3)$ (iii) $(2,3)$ (iv) $\left(\frac{1}{2},\frac{1}{3}\right)$

2. (i) 23 , 7 (ii) ₹18,000 , ₹14,000 (iii) 42 (iv) ₹800 (v) 253cm^2 (vi) 720 km

Exercise 3.3

1. (i) $4,-2$ (ii) $\frac{1}{2},\frac{1}{2}$ (iii) $\frac{3}{2}, -\frac{1}{3}$ (iv) $0,-2$

(v) $\sqrt{15}, -\sqrt{15}$ (vi) $\frac{2}{3},1$ (vii) $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ (viii) $-13, 11$

2. (i) x^2-3x+1 (ii) x^2-2x+4 (iii) x^2+4 (iv) $x^2-\sqrt{2}x+\frac{1}{5}$

(v) $x^2-\frac{x}{3}+1$ (vi) $x^2-\frac{x}{2}-4$ (vii) $x^2-\frac{x}{3}-\frac{1}{3}$ (viii) $x^2-\sqrt{3}x+2$

Exercise 3.4

1. (i) $x^2+2x-1,\ 4$ (ii) $3x^2-11x+40,\ -125$ (iii) $x^2+2x-2,\ 2$

(iv) $x^2-\frac{5}{3}x+\frac{5}{9},\ -\frac{50}{9}$ (v) $2x^3-\frac{x^2}{2}-\frac{3}{8}x+\frac{51}{32},\ -\frac{211}{32}$

(vi) $x^3-3x^2-8x+\frac{55}{2},\ -\frac{41}{2}$

2. $a=-6,\ b=11$, Remainder is 5 **3.** $p=-2,\ q=0$, Remainder is -10

Exercise 3.5

1. (i) $(x-1)(x+2)(x-3)$ (ii) $(x-1)(2x+3)(2x-1)$ (iii) $(x-1)(x-12)(x-10)$

(iv) $(x-1)(4x^2-x+6)$ (v) $(x-1)(x-2)(x+3)$ (vi) $(x+1)(x+2)(x+10)$

(vii) $(x-2)(x-3)(2x+1)$ (viii) $(x-1)(x^2+x-4)$ (ix) $(x-1)(x+1)(x-10)$

(x) $(x-1)(x+6)(2x+1)$ (xi) $(x-2)(x^2+3x+7)$ (xii) $(x+2)(x-3)(x-4)$

Exercise 3.6

1. (i) $7x^2yz^3$ (ii) x^2y (iii) $5c^3$ (iv) $7xyz^2$

2. (i) $c-d$ (ii) $x-3a$ (iii) $m+3$ (iv) $x+11$ (v) $x+2y$

(vi) $2x+1$ (vii) $x-2$ (viii) $(x-1)(x^2+1)$ (ix) $4x^2(2x+1)$ (x) $(a-1)^3(a+3)^2$

3. (i) x^2-4x+3 (ii) $x+1$ (iii) $2(x^2+1)$ (iv) x^2+4

Exercise 3.7

1. x^3y^2z **2.** $12x^3y^3z$ **3.** $a^2b^2c^2$ **4.** $264a^4b^4c^4$ **5.** a^{m+3}

6. $xy(x+y)$ **7.** $6(a-1)^2(a+1)$ **8.** $10xy(x+3y)(x-3y)(x^2-3xy+9y^2)$

9. $(x+4)^2(x-3)^3(x-1)$ **10.** $420x^3(3x+y)^2(x-2y)(3x+1)$

Exercise 3.8

1. (i) $(x-3)(x-2)(x+6)$ (ii) $(x^2+2x+3)(x^4+2x^2+x+2)$

(iii) $(2x^2+x-5)(x^3+8x^2+4x-21)$ (iv) $(x^3-5x-8)(2x^3-3x^2-9x+5)$

2. (i) $(x+1)(x+2)^2$ (ii) $(3x-7)^3(4x+5)$ (iii) $(x^2-y^2)(x^4+x^2y^2+y^4)$

(iv) $x(x+2)(5x+1)$ (v) $(x-2)(x-1)$ (vi) $2(x+1)(x+2)$

Exercise 3.9

1. (i) $\dfrac{2x+3}{x-4}$ (ii) $\dfrac{1}{x^2-1}$ (iii) $(x-1)$ (iv) $\dfrac{x^2+3x+9}{x+3}$

(v) x^2-x+1 (vi) $\dfrac{x+2}{x^2+2x+4}$ (vii) $\dfrac{x-1}{x+1}$ (viii) $(x+3)$

(ix) $\dfrac{(x-1)}{(x+1)}$ (x) 1 (xi) $\dfrac{(x+1)}{(2x-1)}$ (xii) $(x-2)$

Exercise 3.10

1. (i) $3x$ (ii) $\dfrac{x+9}{x-2}$ (iii) $\dfrac{1}{x+4}$ (iv) $\dfrac{1}{x-1}$ (v) $\dfrac{2x+1}{x+2}$ (vi) 1

2. (i) $\dfrac{x-1}{x}$ (ii) $\dfrac{x-6}{x-7}$ (iii) $\dfrac{x+1}{x-5}$ (iv) $\dfrac{x-5}{x-11}$ (v) 1 (vi) $\dfrac{3x+1}{4(3x+4)}$ (vii) $\dfrac{x-1}{x+1}$

Exercise 3.11

1. (i) $x^2 + 2x + 4$ (ii) $\frac{2}{x+1}$ (iii) $\frac{2(x+4)}{x+3}$ (iv) $\frac{2}{x-5}$

   (v) $\frac{x+1}{x-2}$ (vi) $\frac{4}{x+4}$ (vii) $\frac{2}{x+1}$ (viii) 0

2. $\frac{2x^3+2x^2+5}{x^2+2}$ 3. $\frac{5x^2-7x+6}{2x-1}$ 4. 1

Exercise 3.12

1. (i) $14|a^3b^4c^5|$ (ii) $17|(a-b)^2(b-c)^3|$ (iii) $|x-11|$

   (iv) $|x+y|$ (v) $\frac{11}{9}\left|\frac{x^2}{y}\right|$ (vi) $\frac{8}{5}\left|\frac{(a+b)^2(x-y)^4(b-c)^3}{(x+y)^2(a-b)^3(b+c)^5}\right|$

2. (i) $|4x-3|$ (ii) $|(x+5)(x-5)(x+3)|$ (iii) $|2x-3y-5z|$

   (iv) $\left|x^2+\frac{1}{x^2}\right|$ (v) $|(2x+3)(3x-2)(2x+1)|$ (vi) $|(2x-1)(x-2)(3x+1)|$

Exercise 3.13

1. (i) $|x^2-2x+3|$ (ii) $|2x^2+2x+1|$ (iii) $|3x^2-x+1|$ (iv) $|4x^2-3x+2|$
2. (i) $a=-42, b=49$ (ii) $a=12, b=9$ (iii) $a=49, b=-70$ (iv) $a=9,\ b=-12$

Exercise 3.14

1. $\{-6,3\}$ 2. $\left\{-\frac{4}{3},3\right\}$ 3. $\left\{-\sqrt{5},\frac{3}{\sqrt{5}}\right\}$ 4. $\left\{-\frac{3}{2},5\right\}$ 5. $\left\{-\frac{4}{3},2\right\}$

6. $\left\{5,\frac{1}{5}\right\}$ 7. $\left\{-\frac{5}{2},\frac{3}{2}\right\}$ 8. $\left\{\frac{1}{b^2},\frac{1}{a^2}\right\}$ 9. $\left\{-\frac{5}{2},3\right\}$ 10. $\left\{7,\frac{8}{3}\right\}$

Exercise 3.15

1. (i) $\{-7,1\}$ (ii) $\left\{\frac{-3+\sqrt{5}}{2},\frac{-3-\sqrt{5}}{2}\right\}$ (iii) $\left\{-3,\frac{1}{2}\right\}$

   (iv) $\left\{\frac{a-b}{2},-\left(\frac{a+b}{2}\right)\right\}$ (v) $\{\sqrt{3},1\}$ (vi) $\{-1,3\}$

2. (i) $\{4,3\}$ (ii) $\left\{\frac{2}{5},\frac{1}{3}\right\}$ (iii) $\left\{\frac{1}{2},2\right\}$ (iv) $\left\{-\frac{2b}{3a},\frac{b}{a}\right\}$

   (v) $\left\{\frac{1}{a},a\right\}$ (vi) $\left\{\frac{a+b}{6},\frac{a-b}{6}\right\}$ (vii) $\left\{\frac{(9+\sqrt{769})}{8},\frac{(9-\sqrt{769})}{8}\right\}$ (viii) $\left\{-1,\frac{b^2}{a^2}\right\}$

Exercise 3.16

1. 8 or $\frac{1}{8}$ 2. 9 and 6 3. 20 m, 5m or 10m, 10m 4. $\frac{3}{2}m$
5. 45km/hr 6. 5 km/hr 7. 49 years, 7 years 8. 24 cm 9. 12 days
10. Speed of the first train = 20 km / hr and the speed of the second train = 15 km / hr

Exercise 3.17

1. (i) Real (ii) Non-real (iii) Real and equal (iv) Real and equal (v) Non-real (vi) Real
2. (i) $\frac{25}{2}$ (ii) ± 3 (iii) -5 or 1 (iv) 0 or 3

Exercise 3.18

1. (i) 6,5 (ii) $-\frac{r}{k}, p$ (iii) $\frac{5}{3}, 0$ (iv) $0, -\frac{25}{8}$
2. (i) $x^2 - 7x + 12 = 0$ (ii) $x^2 - 6x + 2 = 0$ (iii) $4x^2 - 16x + 9 = 0$
3. (i) $\frac{13}{6}$ (ii) $\pm\frac{1}{3}$ (iii) $\frac{35}{18}$ **4.** $\frac{4}{3}$
5. $4x^2 - 29x + 25 = 0$ **6.** $x^2 + 3x + 2 = 0$ **7.** $x^2 - 11x + 1 = 0$
8. (i) $x^2 - 6x + 3 = 0$ (ii) $27x^2 - 18x + 1 = 0$ (iii) $3x^2 - 18x + 25 = 0$
9. $x^2 + 3x - 4 = 0$ **10.** $k = -18$ **11.** $a = \pm 24$ **12.** $p = \pm 3\sqrt{5}$

Exercise 3.19

1	2	3	4	5	6	7	8	9	10
B	C	A	A	C	D	B	C	C	C
11	12	13	14	15	16	17	18	19	20
D	B	A	A	A	D	D	D	B	C
21	22	23	24	25					
D	A	C	C	A					

4. MATRICES

Exercise 4.1

1. $\begin{pmatrix} 400 & 500 \\ 200 & 250 \\ 300 & 400 \end{pmatrix}$, $\begin{pmatrix} 400 & 200 & 300 \\ 500 & 250 & 400 \end{pmatrix}$, 3×2 , 2×3 **2.** $\begin{pmatrix} 6 \\ 8 \\ 13 \end{pmatrix}$, $(6 \;\; 8 \;\; 13)$
3. (i) 2×3 (ii) 3×1 (iii) 3×3 (iv) 1×3 (v) 4×2
4. $1\times 8,\ 8\times 1,\ 2\times 4,\ 4\times 2$
5. $1\times 30, 30\times 1, 2\times 15, 15\times 2, 3\times 10, 10\times 3, 5\times 6, 6\times 5.$
6. (i) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ **7.** (i) $\begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 1 \\ 3 & \frac{3}{2} \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & \frac{2}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (iii) $\begin{pmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \\ \frac{3}{2} & 0 \end{pmatrix}$
8. (i) 3×4 (ii) 4, 0 (iii) 2^{nd} row and 3^{rd} column **9.** $A^T = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 1 & 0 \end{pmatrix}$

Exercise 4.2

1. $x = 2,\ y = -4,\ z = -1$ 2. $x = 4,\ y = -3$

3. $\begin{pmatrix} -1 & 2 \\ 16 & -6 \end{pmatrix}$ 4. $\begin{pmatrix} 14 & 3 \\ 14 & 5 \end{pmatrix}$ 5. $\begin{pmatrix} 0 & -18 \\ 33 & -45 \end{pmatrix}$ 6. $a = 3,\ b = -4$

7. $X = \begin{pmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{pmatrix}$, $Y = \begin{pmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{pmatrix}$ 8. $x = -3, -3,\ y = -1, 4$

11. $\begin{array}{ccccl} \text{TV} & \text{DVD} & \text{Video} & \text{CD} & \\ \end{array}$
$\begin{pmatrix} 55 & 27 & 20 & 16 \\ 72 & 30 & 25 & 27 \\ 47 & 33 & 18 & 22 \end{pmatrix} \begin{array}{l} \text{store I} \\ \text{store II} \\ \text{store III} \end{array}$

12. $\begin{array}{cc} \text{child} & \text{adult} \end{array}$
$\begin{pmatrix} 5 & 5 \\ 10 & 10 \end{pmatrix} \begin{array}{l} \text{Before 2.00p.m.} \\ \text{After 2.00p.m.} \end{array}$

Exercise 4.3

1. (i) 4×2 (ii) not defined (iii) 3×5 (iv) 2×2

2. (i) (6) (ii) $\begin{pmatrix} 8 & -11 \\ 22 & 12 \end{pmatrix}$ (iii) $\begin{pmatrix} -40 & 64 \\ 22 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 12 & -42 \\ -6 & 21 \end{pmatrix}$

3. $\begin{pmatrix} 1750 \\ 1600 \\ 1650 \end{pmatrix} \begin{array}{l} \text{I day} \\ \text{II day} \\ \text{III day} \end{array}$, (5000) 4. $x = 3,\ y = 0$ 5. $x = 2,\ y = -5$

7. $AB = \begin{pmatrix} 15 & 4 \\ 12 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 9 & 6 \\ 17 & 6 \end{pmatrix}$, $AB \neq BA$ 11. $x = -3, 5$

Exercise 4.4

1	2	3	4	5	6	7	8	9	10
D	D	A	D	B	D	B	C	C	A
11	12	13	14	15	16	17	18	19	20
B	D	D	B	C	B	A	C	B	D

5. COORDINATE GEOMETRY

Exercise 5.1

1. (i) $(-2, 1)$ (ii) $(0,2)$ 2. (i) $(5,-2)$ (ii) $(2, -1)$ 3. $(-12, 8)$

4. $(2, -2)$ 6. $(-24,-2)$ 7. $(-2,3)$ 8. $(-6,-3)$ 9. $(-1,0), (-4,2)$

10. $\left(-3, \frac{3}{2}\right)$, $(-2, 3)$, $\left(-1, \frac{9}{2}\right)$ 11. 4 : 7 internally

12. 5:2 internally, $\left(0, \frac{17}{7}\right)$ 13. $\frac{\sqrt{130}}{2}$, $\sqrt{13}$, $\frac{\sqrt{130}}{2}$

Exercise 5.2

1. (i) 3 sq. units (ii) 32 sq. units (iii) 19 sq. units

2. (i) $a = -3$ (ii) $a = \frac{13}{2}$ (iii) $a = 1, 3$

3. (i) collinear (ii) not collinear (iii) collinear

4. (i) k = 1 (ii) k = 2 (iii) $k = \frac{7}{3}$

5. (i) 17 sq. units (ii) 43 sq. units (iii) 60.5 sq. units 7. 1 sq. units, 1 : 4

Exercise 5.3

1. (i) 45° (ii) 60° (iii) 0° 2. (i) $\frac{1}{\sqrt{3}}$ (ii) $\sqrt{3}$ (iii) undefined

3. (i) 1 (ii) –2 (iii) 1 4. (i) 45° (ii) 30° (iii) $\tan\theta = \frac{b}{a}$

5. $-\frac{1}{2}$ 6. (i) 0 (ii) undefined (iii) 1 7. $\sqrt{3}$, 0 10. $a = -1$

11. $b = 6$ 12. $-\frac{9}{10}$ 13. $\frac{11}{7}, -13, -\frac{1}{4}$ 14. $\frac{1}{12}, -\frac{4}{5}, \frac{9}{2}$

Exercise 5.4

1. $y = 5$, $y = -5$ 2. $y = -2$, $x = -5$ 3. (i) $3x + y - 4 = 0$ (ii) $\sqrt{3}x - y + 3 = 0$

4. $x - 2y + 6 = 0$ 5. (i) slope 1, y-intercept 1 (ii) slope $\frac{5}{3}$, y-intercept 0

(iii) slope 2, y-intercept $\frac{1}{2}$ (iv) slope $-\frac{2}{3}$, y-intercept $-\frac{2}{5}$

6. (i) $4x + y - 6 = 0$ (ii) $2x - 3y - 22 = 0$ 7. $2x - 2\sqrt{3}y + (3\sqrt{3} - 7) = 0$

8. (i) $x - 5y + 27 = 0$ (ii) $x + y + 6 = 0$ 9. $6x + 5y - 2 = 0$

11. (i) $3x + 2y - 6 = 0$ (ii) $9x - 2y + 3 = 0$ (iii) $15x - 8y - 6 = 0$

12. (i) 3,5 (ii) –8, 16 (iii) $-\frac{4}{3}, -\frac{2}{5}$, 13. $2x + 3y - 18 = 0$

14. $2x + y - 6 = 0$, $x + 2y - 6 = 0$ 15. $x - y - 8 = 0$

16. $x + 3y - 6 = 0$ 17. $2x + 3y - 12 = 0$ 18. $x + 2y - 10 = 0$, $6x + 11y - 66 = 0$

19. $x + y - 5 = 0$ 20. $3x - 2y + 4 = 0$

Exercise 5.5

1. (i) $-\frac{3}{4}$ (ii) 7 (iii) $\frac{4}{5}$ 4. $a = 6$ 5. $a = 5$ 6. $p = 1,2$ 7. $h = \frac{22}{9}$

8. $3x - y - 5 = 0$ 9. $2x + y = 0$ 10. $2x + y - 5 = 0$ 11. $x + y - 2 = 0$

12. $5x + 3y + 8 = 0$ 13. $x + 3y - 7 = 0$ 14. $x - 3y + 6 = 0$

15. $x - 4y + 20 = 0$ 16. (3, 2) 17. 5 units 18. $x + 2y - 5 = 0$

19. $2x + 3y - 9 = 0$

Exercise 5.6

1	2	3	4	5	6	7	8	9	10	11	12
C	B	A	D	A	B	D	A	D	C	C	B
13	14	15	16	17	18	19	20	21	22	23	
C	C	C	D	B	B	D	A	A	B	B	

6. GEOMETRY

Exercise 6.1

1. (i) 20cm (ii) 6cm (iii) 1   **2.** 7.5cm   **3.** (i) No (ii) Yes   **4.** 10.5cm

6. 12cm, 10cm   **9.** (i) 7.5cm (ii) 5.8cm (iii) 4 cm   **10.** (i) Yes (ii) No   **11.** 18 cm

Exercise 6.2

1. (i) $x = 4\text{cm},\ y = 9\text{cm}$ (ii) $x = 3.6\text{cm},\ y = 2.4\text{cm},\ z = 10\text{cm}$ (iii) $x = 8.4\text{cm},\ y = 2.5\text{cm}$

2. 3.6m   **3.** 1.2m   **4.** 140m   **6.** 6cm   **7.** 64cm^2 **8.** 166.25 cm

9. (i) $\frac{9}{64}$ (ii) $\frac{55}{64}$ **10.** 6.3km^2 **11.** 72cm   **12.** 9m

13. (i) $\triangle XWY,\ \triangle YWZ,\ \triangle XYZ$ (ii) 4.8m

Exercise 6.3

1. 65° **2.** (i) 4 cm (ii) 12 cm   **3.** (i) 12 cm (ii) 5 cm   **6.** 30 cm

Exercise 6.4

1	2	3	4	5	6	7	8	9	10
B	B	A	D	B	C	B	D	B	B
11	12	13	14	15	16	17	18	19	20
D	D	C	D	D	A	B	B	D	C

7. TRIGONOMETRY

Exercise 7.1

1. (i) No (ii) No

Exercise 7.2

1. 1.8m   **2.** 30° **3.** No   **4.** 174.7 m   **5.** 40 cm   **6.** Crow B

7. $5\sqrt{6}$ m   **8.** 1912.40 m   **9.** $30\sqrt{2}$ m   **10.** 1.098 m   **11.** $19\sqrt{3}$ m

12. Yes   **13.** 87m   **14.** 3 Minutes   **15.** 3464 km   **16.** 40 m

17. 60 m ; $40\sqrt{3}$ m   **18.** 90m

Exercise 7.3

1	2	3	4	5	6	7	8	9	10
B	C	C	A	A	B	A	A	C	B
11	12	13	14	15	16	17	18	19	20
B	C	A	D	C	C	D	B	B	D

8. MENSURATION

Exercise 8.1

1. 704cm^2, $1936\,\text{cm}^2$
2. $h = 8$ cm, 352cm^2
3. $h = 40$ cm, d = 35 cm
4. ₹2640
5. $r = 3.5$ cm, h = 7 cm
6. $h = 28$ cm
7. $C_1 : C_2 = 5 : 2$
8. $1300\pi\,\text{cm}^2$
9. 3168 cm^2
10. 550cm^2 , 704 cm^2
11. $h = 15\sqrt{3}$ cm, $l = 30$ cm
12. 1416cm^2
13. 23.1m^2
14. 10.5 cm
15. $301\frac{5}{7}\text{ cm}^2$
16. 2.8 cm
17. 4158cm^2
18. $C_1 : C_2 = 9 : 25$, $T_1 : T_2 = 9 : 25$
19. $44.1\pi\text{ cm}^2$, $57.33\pi\text{ cm}^2$
20. ₹246 .40

Exercise 8.2

1. 18480 cm^3
2. 38.5 litres
3. 4620 cm^3
4. $r = 2.1$ cm
5. $V_1 : V_2 = 20 : 27$
6. 10 cm
7. 4158 cm^3
8. 7.04 cm^3
9. 8800cm^3
10. 616cm^3
11. 5cm
12. 1408.6 cm^3
13. $314\frac{2}{7}\text{cm}^3$
14. $2\sqrt{13}$ cm
15. 8 cm
16. 2.29 Kg
17. $3050\frac{2}{3}\text{cm}^3$
18. $288\pi\text{cm}^2$
19. $718\frac{2}{3}\text{cm}^3$
20. 1: 8

Exercise 8.3

1. $11.88\pi\text{ cm}^2$
2. 7623cm^3
3. 220mm^2
4. 1034 sq.m
5. 12 cm
6. 12.8 km
7. 2 cm
8. 1 cm
9. 1386 litres
10. 3 hrs. 12 mins.
11. 16 cm
12. 16 cm
13. 750 lead shots
14. 10 cones
15. 70 cm
16. $r = 36$ cm, $l = 12\sqrt{13}$ cm
17. 11m

Exercise 8.4

1	2	3	4	5	6	7	8	9	10	11
B	C	A	A	B	C	A	B	D	C	C
12	13	14	15	16	17	18	19	20	21	22
D	D	B	D	B	C	B	D	A	D	C

10. GRAPH

Exercise 10.1

2. (i) $\{-2, 2\}$ (ii) $\{-2, 5\}$ (iii) $\{5, 1\}$ (iv) $\left\{-\frac{1}{2}, 3\right\}$
3. $\{-1, 5\}$
4. $\{-2, 3\}$
5. $\{-2.5, 2\}$
6. $\{-3, 5\}$
7. No real solutions

Exercise 10.2

1. 120 kms
2. (i) ₹105 (ii) 11 note books
3. (i) $y = 8$ (ii) $x = 6$
4. (i) $k = 15$ (ii) ₹ 45
5. $y = 4$; $x = 2$
6. 24 days

11. STATISTICS

Exercise 11.1

1. (i) 36, 0.44 (ii) 44, 0.64 **2.** 71 **3.** 3.38 kg **4.** $2\sqrt{5}$, 20

5. 3.74 **6.** (i) 5.97 (ii) 4.69 **7.** 6.32 **8.** 1.107 **9.** 15.08

10. 36.76, 6.06 **11.** 416, 20. 39 **12.** 54.19 **13.** 4800, 240400 **14.** 10.2, 1.99

15. 25 **16.** 20.41 **17.** 12 **18.** 5.24 **19.** 1159, 70

20. *A* is more consistent

Exercise 11.2

1	2	3	4	5	6	7	8	9	10
D	A	C	B	D	C	C	B	A	D
11	12	13	14	15					
D	B	C	D	B					

12. PROBABILITY

Exercise 12.1

1. $\frac{1}{10}$ **2.** $\frac{1}{9}$ **3.** $\frac{1}{3}$ **4.** $\frac{1}{5}$ **5.** $\frac{3}{4}$

6. (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$ (iii) $\frac{12}{13}$ **7.** (i) $\frac{7}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$

8. (i) $\frac{1}{2}$ (ii) $\frac{3}{5}$ **9.** (i) $\frac{1}{10}$ (ii) $\frac{24}{25}$ **10.** $\frac{1}{2}$ **11.** (i) $\frac{1}{4}$ (ii) $\frac{2}{3}$

12. (i) $\frac{1}{4}$ (ii) $\frac{17}{20}$ **13.** $\frac{1}{3}$ **14.** $\frac{1}{36}$ **15.** $\frac{1}{6}$ **16.** 12

17. (i) $\frac{22}{25}$ (ii) $\frac{24}{25}$ **18.** (i) $\frac{1}{4}$ (ii) 3 **19.** (i) $\frac{5}{9}$ (ii) $\frac{17}{18}$

Exercise 12. 2

1. $\frac{4}{5}$ **2.** $\frac{3}{20}$ **3.** (i) $\frac{1}{5}$ (ii) $\frac{4}{5}$ **4.** $\frac{5}{9}$ **5.** $\frac{8}{25}$

6. $\frac{5}{8}$ **7.** $\frac{4}{9}$ **8.** $\frac{9}{10}$ **9.** $\frac{3}{5}$ **10.** $\frac{4}{13}$

11. $\frac{8}{13}$ **12.** $\frac{2}{3}$ **13.** $\frac{5}{13}, \frac{4}{13}$ **14.** (i) 0.45 (ii) 0.3 **15.** $\frac{101}{105}$

Exercise 12. 3

1	2	3	4	5	6	7	8	9	10
C	**D**	**B**	**A**	**A**	**B**	**A**	**A**	**D**	**A**
11	12	13	14	15	16	17	18	19	20
D	**C**	**C**	**B**	**B**	**D**	**D**	**A**	**A**	**B**

Miscellaneous problems

(Not for examination)

1. If $f(x) = \dfrac{x-1}{x+1}, x \neq -1,$ then prove that $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.
2. Solve the equation $(x-1)(x-2)(x-3)(x-4) = 15$ for real values of x.

   $\left(\text{Ans}: x = \dfrac{5 \pm \sqrt{21}}{2}\right)$
3. For what values of x do the three numbers $\log_{10}2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ taken in that order constitute an A.P.? $(\text{Ans}: x = \log_5 2)$
4. In a G.P. with common ratio r, the sum of first four terms is equal to 15 and the sum of their squares is equal to 85. Prove that $14r^4 - 17r^3 - 17r^2 - 17r + 14 = 0$.
5. Prove that the sequence $\{b_n\}$ is a G.P. if and only if $b_n^2 = b_{n-1}\, b_{n+1}, n > 1$.
6. Certain numbers appear in both arithmetic progressions 17, 21, $\cdots$ and 16, 21, $\cdots$. Find the sum of the first ten numbers appearing in both progressions.
7. Prove that the sequence $\{a_n\}$ is an A.P. if and only if $a_n = \dfrac{a_{n-1} + a_{n+1}}{2}, n > 1$.
8. Prove that $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$
9. Prove that $\dfrac{\sin x + \cos x}{\cos^2 x} = \tan^3 x + \tan^2 x + \tan x + 1$.
10. If we divide a two-digit number by the sum of its digits, we get 4 as a quotient and 3 as a remainder. Now if we divide that two-digit number by the product of its digits, we get 3 as a quotient and 5 as a remainder. Find the two-digit number. (Ans : 23)
11. Find the sum of all two-digit numbers which, being divided by 4, leave a remainder of 1. (Ans : 1210)
12. Simplify the expression $\dfrac{\dfrac{1}{a} + \dfrac{1}{b+c}}{\dfrac{1}{a} - \dfrac{1}{b+c}} \times \left(1 + \dfrac{b^2 + c^2 - a^2}{2bc}\right)(a+b+c)^{-2}$

    $\left(\text{Ans}: \dfrac{1}{2bc}\right)$
13. The quadratic equation $ax^2 + bx + c = 0$ has no real roots and $a + b + c < 0$. Find the sign of the number c. (Hint. If $f(x) = 0$ has no real roots, then $f(x)$ has same sign for all x) (Ans: $c < 0$)
14. Find all real numbers x such that $f(x) = \dfrac{x-1}{x^2 - x + 6} > 0$. (Ans $x > 1$)
15. Solve the equation $1 + a + a^2 + \cdots + a^x = (1+a)(1+a^2)(1+a^4)(1+a^8)$

    (Ans: $x = 15$)
16. Compute $\dfrac{6x_1^2x_2 - 4x_1^3 + 6x_1x_2^2 - 4x_2^3}{3x_1^2 + 5x_1x_2 + 3x_2^2}$, where x_1 and x_2 are the roots of the equation $x^2 - 5x + 2 = 0$. $\left(\text{Ans}: -\dfrac{320}{73}\right)$
17. Prove the identity: $\operatorname{cosec}\alpha - \cot\alpha - \dfrac{\sin\alpha + \cos\alpha}{\cos\alpha} + \dfrac{\sec\alpha - 1}{\sin\alpha} = -1$

18. One-fourths of a herd of camels was seen in the forest. Twice the square root of the number of herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels. (Ans: Number of camels is 36)

19. After covering a distance of 30 km with a uniform speed there is some defect in a train engine and therefore, its speed is reduced to $\frac{4}{5}$ of its original speed. Consequently, the train reaches its destination late by 45 minutes. Had it happened after covering 18 kilo metres more, the train would have reached 9 minutes earlier. Find the speed of the train and the distance of journey. (Ans: Speed of the train is 30 km/hr and the distance of the journey is 120 km.)

20. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

21. If $\operatorname{cosec}\theta - \sin\theta = l$ and $\sec\theta - \cos\theta = m$, prove that $l^2m^2(l^2 + m^2 + 3) = 1$

22. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain. (Ans: 1.366 km)

23. If the opposite angular points of a square are (3, 4) and (1, –1), then find the coordinates of the remaining angular points. (Ans: $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$)

24. In an increasing G.P. the sum of first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression. (Ans : 6)

25. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a height b just above A is β. Prove that the height of the tower is $b\cot\beta\tan\alpha$.

26. A rectangular pool has the dimensions 40 ft × 20 ft. We have exactly 99 cu.ft of concrete to be used to create a border of uniform width and depth around the pool. If the border is to have a depth of 3 inches and if we use all of the concrete, how wide the border will be ? (Ans : 3 ft)

27. Simplify $(1 + \frac{2}{2})(1 + \frac{2}{3})(1 + \frac{2}{4})\cdots(1 + \frac{2}{n})$. (Ans : $\frac{(n+1)(n+2)}{6}$)

28. There are three circular disks such that two of them has radius r inches and the third has radius $2r$ inches. These three disks are placed in a plane such that each of its boundary has exactly one point in common with any other boundary. Find the area of the triangle formed by the centers of these disks. (Ans : $2\sqrt{2}\,r^2$ sq.inches)

29. Six circular discs each having radius 8 inches are placed on the floor in a circular fashion so that in the center area we could place a seventh disk touching all six of these disks exactly at one point each and each disk is touching two other disks one point each on both sides. Find the area formed by these six disks in the center. (Ans : $192\sqrt{3}$ sq. inches)

30. From a cylinderical piece of wood of radius 4 cm and height 5cm, a right circular cone with same base radius and height 3 cms is carved out.Prove that the total surface area of the remaining wood is 76π cm^2.

31. Show that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ where $n! = 1\times2\times3\times\cdots\times n$.

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QUESTION PAPER DESIGN

Subject : Mathematics | **Time: 2.30 Hrs**

Class : X | **Max marks: 100**

Weightage of marks to Learning Objectives

Objectives	Percentage
Knowledge	19
Understanding	31
Application	23
Skill	27
Total	100

Weightage to the types of Question

Type of Questions	Section-A Very Short Answer (Objective)	Section-B Short Answer	Section-C Long Answer	Section-D Very Long Answer	Total
Number of Questions	15	10	9	2	36
Marks	15	20	45	20	100
Time (in minutes)	20	35	65	30	2.30 Hrs

Difficulty Level

Level	Percentage of Marks
Difficult	12
Average	28
Easy	60

Sections and Options

<table>
<tr><th rowspan="2">Sections</th><th colspan="2">Question numbers</th><th rowspan="2">Number of Questions</th><th rowspan="2">Questions to be answered</th></tr>
<tr><th>From</th><th>To</th></tr>
<tr><td>A</td><td>1</td><td>15</td><td>15</td><td>15</td></tr>
<tr><td>B</td><td>16</td><td>30</td><td>16
30th Question is compulsory and is in ‘either’ ‘or’ type</td><td>10</td></tr>
<tr><td>C</td><td>31</td><td>45</td><td>16
45th Question is compulsory and is in ‘either’ ‘or’ type</td><td>9</td></tr>
<tr><td rowspan="2">D</td><td colspan="2">46</td><td>2
This Question is in ‘either’ ‘or’ type</td><td>1</td></tr>
<tr><td colspan="2">47</td><td>2
This Question is in ‘either’ ‘or’ type</td><td>1</td></tr>
</table>

Weightage to Content

Chapter No.	Chapter	Number of Questions				Total Marks
		1 mark	2 marks	5 marks	10 marks	
1	Sets and Functions	1	2	2		15
2	Sequences and series of Real Numbers	2	1	2		14
3	Algebra	2	2	3		21
4	Matrices	1	2	1		10
5	Coordinate Geometry	2	2	2		16
6	Geometry	2	1	1		9
7	Trigonometry	2	2	1		11
8	Mensuration	1	2	2		15
9	Practical Geometry				2	20
10	Graphs				2	20
11	Statistics	1	1	1		8
12	Probability	1	1	1		8
Total		15	16	16	4	167

Distribution of Marks and Questions towards Examples, Exercises and Framed questions

	Sec A (1 mark)	Sec B (2 marks)	Sec C (5 marks)	Sec D (10 marks)	Total Marks	Percentage
From the Examples given in the Text Book	---	6 (2)	6 (5)	1 (10)	52	31
From the Exercises given in the Text Book	10 (1)	8 (2)	8 (5)	3 (10)	96	58
Framed questions from specified chapters	5 (1)	2 (2)	2 (5)	---	19	11
Total	15 (1)	16 (2)	16 (5)	4 (10)	167	100

- Numbers in brackets indicate the marks for each question.

Section - A

1. All the 15 questions numbered 1 to 15 are multiple choice questions each with 4 distractors and all are compulsory. Each question carries one mark.
2. Out of 15 questions, 10 questions are from the multiple choice questions given in the Text Book. The remaining 5 questions should be framed from the five different chapters 2, 3, 5, 6 and 7 on the basis of the Text Book theorems, results, examples and exercises.

Section - B

1. 10 questions are to be answered from the questions numbered 16 to 30. Each question carries two marks.
2. Answer any 9 questions from the first 14 questions. Question No. 30 is compulsory and is in either or type.
3. The order of the first 14 questions should be in the order of the chapters in the Text Book.
4. Out of first 14 questions, 6 questions are from the examples and 8 questions are from the exercises.
5. The two questions under question no. 30 should be framed based on the examples and problems given in the exercises from any two different chapters of 2, 3, 5 and 8.

Section - C

1. 9 questions are to be answered from the questions numbered 31 to 45. Each question carries five marks.
2. Answer any 8 questions from the first 14 questions. Question no. 45 is compulsory and is in either or type.
3. The order of the first 14 questions should be in the order of the chapters in the Text Book.
4. Out of first 14 questions, 6 questions are from the examples and 8 questions are from the exercises.
5. The two questions under question no. 45 should be framed based on the examples and problems given in the exercises from any two different chapters of 2, 3, 5 and 8.
6. Questions numbered 30(a), 30(b), 45(a) and 45(b) should be framed based on the examples and problems given in the exercises from the chapters 2, 3, 5 and 8 subject to the condition that all of them should be from different chapters.

Section - D

1. This section contains two questions numbered 46 and 47, one from the chapter 9 and the other from the chapter 10, each with two alternatives ('either' 'or' type) from the same chapter. Each question carries ten marks.
2. Answer both the questions.
3. One of the questions 46(a), 47(a), 46(b) and 47(b) should be from the examples given in the text book. The remaining three questions should be from the exercises.

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Chapter / Objective	Knowledge				Understanding				Application				Skill				Total marks
	VSA	SA	LA	VLA	VSA	SA	LA	VLA	VSA	SA	LA	VLA	VSA	SA	LA	VLA	
Sets and Functions	1(1)	2(1)	5(1)			2(1)					5(1)						15
Sequences and Series of Real Numbers		2(1)	5(1)		1(1)				1(1)		5(1)						14
Algebra		2(1)	5(1)		1(1)				1(1)	2(1)	5(1)				5(1)		21
Matrices						4(2)	5(1)		1(1)								10
Coordinate Geometry		2(1)			1(1)	2(1)	5(1)		1(1)		5(1)						16
Geometry					1(1)	2(1)	5(1)		1(1)								9
Trigonometry					1(1)	2(1)	5(1)		1(1)	2(1)							11
Mensuration	1(1)					2(1)	5(1)			2(1)	5(1)						15
Practical Geometry																10(2)	20
Graphs																10(2)	20
Statistics			5(1)			2(1)			1(1)								8
Probability		2(1)					5(1)		1(1)								8
Total	2(2)	10(5)	20(4)		5(5)	16(8)	30(6)		8(8)	6(3)	25(5)				5(1)	40(4)	167

- Numbers in brackets indicate the number of questions.
- Other numbers indicate the marks.