REVIEW TEAM: 2016-17

MATHEMATICS: CLASS XII

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MATHEMATICS CLASS 12 SYLLABUS

Course Structure

Unit	Topic	Marks
I.	Relations and Functions	10
II.	Algebra	13
III.	Calculus	44
IV.	Vectors and 3-D Geometry	17
V.	Linear Programming	06
VI.	Probability	10
	Total :	**100**

Unit I: RELATIONS AND FUNCTIONS

1. **Relations and Functions**

   Types of Relations: Reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of a function. Binary operations.

2. **Inverse Trigonometric Functions**

   Definition, range, domain, principal value branch. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

Unit II: ALGEBRA

1. Matrices

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices.

Operation on Matrices: Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). Concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. Determinants

Determinant of a square matrix (up to 3×3 matrices), properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle. adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Unit III : CALCULUS

1. Continuity and Differentiability

Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit function. Concept of exponential and logarithmic functions.

Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's mean Value Theorems (without proof) and their geometric interpretations.

2. Applications of Derivatives

Applications of Derivatives : Rate of change of bodies, increasing/ decreasing functions, tangents and normal's, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

3. Integrals

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts. Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int \sqrt{a^2 \pm x^2}dx, \int \sqrt{x^2 - a^2}dx,$$

$$\int \sqrt{ax^2+bx+c}\, dx, \int (px+q)\sqrt{ax^2+bx+c}dx$$

(Definite integrals as a limit of a sum, Fundamental theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

4. Applications of the Integrals

Applications in finding the area under simple curves, especially lines, circles/parabolas/ellipses (in standard form only), area between any of the two above said curves (the region should be clearly identifiable).

5. Differential Equations

Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, Solution of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type:

$dy/dx+py = q$, where p and q are functions of x or constants.

$dx/dy+px = q$, where p and q are functions of y or constants

Unit IV: VECTORS AND THREE-DIMENSIONAL GEOMETRY

1. Vectors

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and applications of scalar (dot) product of vectors, vector (cross) product of vectors, scalar triple product of vectors.

2. Three-Dimensional Geometry

Direction cosines and direction ratios of a line joining two points. Cartesian and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (*i*) two lines, (*ii*) two planes, (*iii*) a line and a plane. Distance of a point from a plane.

Unit V: LINEAR PROGRAMMING

1. **Linear Programming:** Introduction, related terminology such as constraints, objective function, optimization. Different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded and unbounded) feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

Unit VI : PROBABILITY

1. Probability

Conditional probability, Multiplication theorem on probability, independent events, total probability, Baye's theorem, Random variable and its probability distribution, mean and variance of a random variable. Repeated independent (Bernoulli) trials and Binomial distribution.

The following will be applicable in the subject Mathematics (041) for class XII for the academic session 2016-17 and Board examination 2017.

Question Paper Design

S. NO.	Typology of Questions	VSA (1) mark	SA (2) marks	LA- I (4) marks	LA -II (6) marks	Marks	Weightage
1	Remembering	2	2	2	1	20	20%
2	Understanding	1	3	4	2	35	35%
3	Application	1	–	3	2	25	25%
4	HOTS	–	3	1	–	10	10%
5	Evaluation	–	–	1(VBQ)	1	10	10%
	Total	1×4= **4**	2×8= **16**	4×11= **44**	6×6= **36**	100	100%

QUESTION WISE BREAK UP FOR 2016-17

Type of Questions	Marks per Question	Total Number of Questions	Total Marks
VSA	1	4	04
SA	2	8	16
LA - I	4	11	44
LA - II	6	6	36
Total		**29**	**100**

1. No chapter wise weightage. Care to be taken to cover all the chapters.
2. The above template is only a sample. Suitable internal variations may be made for generating similar templates keeping the overall weightage to different form of questions and typology of questions same.

CHAPTER 1

RELATIONS AND FUNCTIONS

IMPORTANT POINTS TO REMEMBER

- Relation R from a set A to a set B is subset of A × B and Relation R in set A is a subset of A × A.
- If $n(A) = r$, $n(B) = s$ from set A to set B then $n(A \times B) = rs$. and number of relations $=2^{rs}$
- $\emptyset$ is also a relation defined on set A, called the void (empty) relation.
- $R = A \times A$ is called universal relation.
- **Reflexive Relation:** Relation R defined on set A is said to be reflexive if $(a, a) \in R\ \forall\ a \in A$.
- **Symmetric Relation :** Relation R defined on set A is said to be symmetric iff $(a, b) \in R \Rightarrow (b, a) \in R\ \forall\ a, b, \in A$
- **Transitive Relation :** Relation R defined on set A is said to be transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R\ \forall\ a, b, c \in A$
- **Equivalence Relation:** A relation defined on set A is said to be equivalence relation iff it is reflexive, symmetric and transitive.
- **Equivalence class of an element:** Let R be an equivalence relation of set A, then equivalence class of $a \in A$ is $[a] = \{ b \in A : (b, a) \in R\}$.
- **One-One Function :** $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B. *i.e.* $\forall\ x_1, x_2 \in A$ such that $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

  OR

  $\forall\ x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

  $$\Rightarrow x_1 = x_2$$

  One-one function is also called injective function.

- Onto function (surjective): A function $f : A \to B$ is said to be onto iff

  $R_f = B$ *i.e.* $\forall\ b \in B$, there exists $a \in A$ such that $f(a) = b$

- **Bijective Function :** A function which is both injective and surjective is called bijective function.

- **Composition of Two Functions :** If $f : A \to B$, $g : B \to C$ are two functions, then composition of f and g denoted by gof is a function from A to C given

  by, $(gof)(x) = g(f(x))\ \forall\ x \in A$

  Clearly gof is defined if Range of $f \subset$ domain of g. Similarly fog can be defined.

- **Invertible Function:** A function $f : X \to Y$ is invertible iff it is bijective.

  If $f : X \to Y$ is bijective function, then function $g : Y \to X$ is said to be inverse of f iff $fog = I_y$ and $gof = I_x$

  when I_x, I_y are identity functions.

- Inverse of f is denoted by f^{-1}. [f^{-1} does not mean $\frac{1}{f}$]

- Let A and B are two nom empty set that $n(A) = \mathrm{p}$ and $n(B) = q$

  Then

  a) Number of functions from A to B $= q^p$

  b) Number of one-one functions from A to B $= \begin{cases} {}^{q}P_{p}, p \le q \\ 0, p > q \end{cases}$

  c) $Number\ of\ onto\ function\ from\ A\ to$ $\begin{cases} \sum_{r=1}^{q} (-1)^{q-r}\ q_{c_r}\ r^p, p \ge q \\ 0 \quad\quad , \quad p < q. \end{cases}$

  d) Number of bijective functions from A to B $= \begin{cases} p!, p = q \\ o, p \ne q. \end{cases}$

- **Binary Operation:** A binary operation '*' defined on set A is a function from $A \times A \to A$.

  *(a, b) is denoted by $a * b$.

- No. of binary operation on set having n elements $= n^{(n^2)}$

- Binary operation * defined on set A is said to be commutative iff

  $a * b = b * a\ \forall\ a, b \in A.$

- Binary operation*defined on set A is called associative iff

  $a * (b * c) = (a * b) * c \ \forall \ a, b, c \in A$

- If * is Binary operation on A, then an element $e \in A$ (if exists) is said to be the identity element iff $a*e = e*a = a \ \forall a \in A$

- Identity element is unique.

- If * is Binary operation on set A, then an element $b \in A$(if exists) is said to be inverse of $a \in A$ iff $a * b = b * a = e$

- Inverse of an element, if it exists, is unique.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. If A is the set of students of a school then write, which of following relations are Universal, Empty or neither of the two.

   $R_1 = \{(a, b) : a, b$ are ages of students and $|a - b| > 0\}$

   $R_2 = \{(a, b) : a, b$ are weights of students, and $|a - b| < 0\}$

   $= \{(a, b) : a, b$ are students studying in same class$\}$

2. Is the relation R in the set $A = \{1, 2, 3, 4, 5\}$ defined as

   $R = \{(a, b) : b = a + 1\}$ reflexive?

3. If R, is a relation in set N given by

   $R = \{(a, b) : a = b - 3, b > 5\}$,

   then does element $(5, 7) \in$ R?

4. If $f : \{1, 3\} \rightarrow \{1, 2, 5\}$ and $g : \{1, 2, 5\} \rightarrow \{1, 2, 3, 4\}$ be given by $f = \{(1, 2), (3, 5)\}$, $g = \{(1, 3), (2, 3), (5, 1)\}$,

   write gof.

5. Let $g, f : R \to R$ be defined by

$g(x) = \frac{x+2}{3}$, $f(x) = 3x - 2$. write fog (x)

6. If $f : R \to R$ defined by

$f(x) = \frac{2x-1}{5}$

be an invertible function, write $f^{-1}(x)$.

7. If $f(x) = \log x$ and $g(x) = e^x$. Find fog and gof, $x > 0$.

8. Let * be a Binary operation defined on R, then if

(i) $a * b = a + b + ab$, write 3 * 2

(ii) $a * b = \frac{(a+b)^2}{3}$, write (2*3)*4.

9. If $n(A) = n(B) = 3$, then how many bijective functions from A to B can be formed?

10. If $f(x) = x + 1$, $g(x) = x - 1$, then (gof) (3) = ?

11. Is $f : N \to N$ given by $f(x) = x^2$ one-one? Give reason.

12. If $f : R \to A$, given by

$f(x) = x^2 - 2x + 2$ is onto function, find set A.

13. If $f : A \to B$ is bijective function such that $n(A) = 10$, then $n(B)$ = ?

14. If $f : R \to R$ defined by $f(x) = \frac{x-1}{2}$, find $(fof)(x)$

15. $R = \{(a, b) : a, b \in N, a \neq b$ and a divides $b\}$. Is R reflexive? Give reason

16. Is $f : R \to R$, given by $f(x) = |x - 1|$ one-one? Give reason

17. $f : R \to B$ given by $f(x) = \sin x$ is onto function, then write set B.

18. If f(x) = log $\left(\frac{1+x}{1-x}\right)$, show that f $\left(\frac{2x}{1+x^2}\right)$ =2f (x).

19. If '*' is a binary operation on set Q of rational numbers given by $a * b = \frac{ab}{5}$ then write the identity element in Q.

20. If * is Binary operation on N defined by $a * b = a + ab \ \forall \ a, b \in N$, write the identity element in N if it exists.

SHORT ANSWER TYPE QUESTIONS (4 Marks)

21. Check the following functions for one-one and onto.

(a) $f : R \rightarrow R$, $f(x) = \frac{2x-3}{7}$

(b) $f : R \rightarrow R$, $f(x) = |x + 1|$

(c) $f : R - \{2\} \rightarrow R$, $f(x) = \frac{3x-1}{x-2}$

(d) $f : R \rightarrow [-1, 1]$, $f(x) = \sin^2 x$

22. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b$ = H.C.F. of a and b. Write the operation table for the operation *.

23. Let $f : R - \left\{\frac{-4}{3}\right\} \rightarrow R - \left\{\frac{4}{3}\right\}$ be a function given by $f(x) = \frac{4x}{3x+4}$.
Show that f is invertible with $f^{-1}(x) = \frac{4x}{4-3x}$

24. Let R be the relation on set $A = \{x : x \in Z, 0 \le x \le 10\}$ given by $R = \{(a, b) : (a - b)$ is divisible by 4\}. Show that R is an equivalence relation. Also, write all elements related to 4.

25. Show that function $f : A \rightarrow B$ defined as $f(x) = \frac{3x+4}{5x-7}$ where $A = R - \left\{\frac{7}{5}\right\}$, $B = R - \left\{\frac{3}{5}\right\}$ is invertible and hence find f^{-1}.

26. Let * be a binary operation on Q such that $a * b = a + b - ab$.

(i) Prove that * is commutative and associative.

(ii) Find identify element of * in Q (if it exists).

27. If * is a binary operation defined on $R - \{0\}$ defined by $a * b = \frac{2a}{b^2}$ then check * for commutativity and associativity.

28. If $A = N \times N$ and binary operation * is defined on A as

$(a, b) * (c, d) = (ac, bd)$.

(i) Check * for commutativity and associativity.

(ii) Find the identity element for * in A (If it exists).

29. Show that the relation R defined by $(a, b) R(c, d) \Leftrightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation.

30. Let * be a binary operation on set Q defined by $a * b = \frac{ab}{4}$, *show that*

(i) 4 is the identity element in Q.

(ii) Every non zero element of Q is invertible with

$a^{-1} = \frac{16}{a}$, $\forall a \in Q - \{0\}$.

31. Show that $f : R_+ \to R_+$ defined by $f(x) = \frac{1}{2x}$ is bijective where R_+ is the set of all non-zero positive real numbers.

32. Test whether relation R defined in R as $R = \{(a-b) : a^2 - 4ab + 3b^2 = 0, a, b \in R\}$ is reflexive, symmetric and transitive.

33. Let $f, g : R \to R$ be two functions defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, then find *gof* and *fog.*

34. Let $A = \{1, 2, 3, ..., 12\}$ and R be a relation in A × A defined by (a, b) R (c, d) if ad = bc $\forall$ (a, b), (c, d) $\in$ A × A. Prove that R is an equivalence relation. Also obtain the equivalence class [(3, 4)].

35. If '*' is a binary operation on R defined by $a * b = a + b + ab$. Prove that * is commutative and associative. Find the identify element. Also show that every element of R is invertible except –1.

36. If f, g : $R \to R$ defined by $f(x) = x^2 - x$ and $g(x) = x + 1$ find (fog) (x) and (gof) (x). Are they equal?

37. f :[1, ∞) → [2, ∞) is given by $f(x) = x + \frac{1}{x}$, find $f^{-1}(x)$.

38. $f : R \to R$, $g : R \to R$ given by $f(x) = [x]$, $g(x) = |x|$ then find

$(fog)\left(\frac{-2}{3}\right)$ and $(gof)\left(\frac{-2}{3}\right)$

39. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad (b + c) = bc (a + d). Show that R is an equivalence relation.

40. Let f : N→R be a function defined as $f(x) = 4x^2 + 12x + 15$.

Show that f : N → S, where S is the range of f, is invertible. Also find the inverse of f. Hence find $f^{-1}(31)$.

41. If the function f : R→ R be defined by f (x) = 2x – 3 and g : R→R by $g(x) = x^3 + 5$, then show that fog is invertible. Also find $(\text{fog})^{-1}(x)$, hence find $(\text{fog})^{-1}(9)$.

42. Let A = $Q \times Q$, where Q is the set of rational number, and * be a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) ∀ (a, b), (c, d) ∈ A.

   i. Is * commutative?
   ii. Is * Associative?
   iii. Find identity element of * in A.
   iv. Find invertible element of A and hence write the inverse of (1,2) and $\left(\frac{1}{3}, -5\right)$

ANSWERS

1. R_1 : is universal relation.

   R_2 : is empty relation.

   R_3 : is neither universal nor empty.

2. No, R is not reflexive.

3. $(5, 7) \notin R$

4. $gof = \{(1, 3), (3, 1)\}$

5. $(fog)(x) = x \ \forall x \in R$

6. $f^{-1}(x) = \frac{5x+1}{2}$

7. $gof\ (x) = x$, $fog, (x) = x$

8. (i) $3 * 2 = 11$ (ii) $\frac{1369}{27}$

9. 6

10. 3

11. Yes, f is one-one $\because \forall\ x_1, x_2 \in N \Rightarrow x_1^2 = x_2^2$.

12. $A = [1, \infty)$ because $R_f = [1, \infty)$

13. $n(B) = 10$

14. $(fof)\ (x) = \frac{x-3}{4}$

15. No, R is not reflexive $\because (a, a) \notin R\ \forall\ a \in N$

16. f is not one-one function

$\because\ f(3) = f(-1) = 2$

$3 \neq -1$ *i.e.* distinct elements have same images.

17. $B = [-1, 1]$

19. $e = 5$

20. Identity element does not exists.

21. (a) Bijective

(b) Neither one-one nor onto.

(c) One-one, but not onto.

(d) Neither one-one nor onto.

22.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

24. Elements related to 4 are 0, 4, 8.

25. $f^{-1}(x) = \frac{7x+4}{5x-3}$

26. 0 is the identity element.

27. Neither commutative nor associative.

28. (i) Commutative and associative.

(ii) (1, 1) is identity in $N \times N$

32. Reflexive, not symmetric, not transitive

33. $gof\,(x) = 0 \ \forall x \in R$,

$fog\,(x) = \begin{cases} 0, x \geq 0 \\ -4x\ x < 0 \end{cases}$

34. [(3, 4)] = {(3, 4) ,(6, 8), (9,12)}

35. *0* is the identity element.

36. $(fog)(x) = x^2 + x$

$(gof)(x) = x^2 - x + 1$

Clearly, they are unequal.

37. $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

38. $(fog)\left(\frac{-2}{3}\right) = 0$

$(gof)\left(\frac{-2}{3}\right) = 1$

40 $f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}$, $f^{-1}(31) = 1$

41. $(fog)^{-1}(x) = \left(\frac{x+7}{2}\right)^{1/3}$, $(fog)^{-1}(9) = 2$

43. I. Not commutative

II. Associative

III. (1,0)

IV. Inverse of (a, b) = $\left(\frac{1}{a}, \frac{-b}{a}\right)$, Inverse of (1, 2) = (1,-2) and Inverse of $\left(\frac{1}{3}, -5\right)$ = (3, 15)

CHAPTER 2

INVERSE TRIGONOMETRIC FUNCTIONS

IMPORTANT POINTS TO REMEMBER

Function	***Domain***	***Range*** ***(Principal Value Branch)***
$\sin^{-1}x$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	[–1, 1]	$[0, \pi]$
$\tan^{-1}x$	*R*	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	*R*	$(0, \pi)$
$\sec^{-1}x$	*R* – (–1, 1)	$[0, \pi]$- $\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1}x$	*R* – (–1, 1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- If sinø = x, ø ∈ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then ø = $\sin^{-1}x$

  If cosø = x, ø ∈ $[0, \pi]$, then ø = $\cos^{-1}x$

  If tanø = x, ø ∈ $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then ø = $\tan^{-1}x$

  If cotø = x, ø ∈ $(0, \pi)$, then ø = $\cot^{-1}x$

  If secø = x, ø ∈ $[0, \pi]$ - $\left\{\frac{\pi}{2}\right\}$, then ø = $\sec^{-1}x$

  If cosecø = x, ø ∈ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$- $\{0\}$, then find ø = $\text{cosec}^{-1}x$

- $\sin^{-1}(\sin x) = x \; \forall \; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  $\cos^{-1}(\cos x) = x \; \forall \; x \in [0, \pi]$

  $\tan^{-1}(\tan x) = x \; \forall \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

  $\cot^{-1}(\cot x) = x \; \forall \; x \in (0, \pi)$

  $\sec^{-1}(\sec x) = x \; \forall \; x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

  $c^{1}(\text{cosec } x) = x \; \forall \; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

- $\sin(\sin^{-1} x) = x \; \forall \; x \in [-1, 1]$

  $\cos(\cos^{-1} x) = x \; \forall \; x \in [-1, 1]$

  $\tan(\tan^{-1} x) = x \; \forall \; x \in R$

  $\cot(\cot^{-1} x) = x \; \forall \; x \in R$

  $\sec(\sec^{-1} x) = x \; \forall \; x \in R - (-1, 1)$

  $\text{cosec}(\text{cosec}^{-1} x) = x \; \forall \; x \in R - (-1, 1)$

- $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right) \; \forall \; x \in [-1, 1]$

  $\tan^{-1} x = \cot^{-1}(1/x) \; \forall \; x > 0$

  $\sec^{-1} x = \cos^{-1}(1/x), \; \forall \; |x| \geq 1$

- $\sin^{-1}(-x) = -\sin^{-1} x \; \forall x \in [-1, 1]$

  $\tan^{-1}(-x) = -\tan^{-1} x \; \forall \; x \in R$

  $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x \; \forall \; |x| \geq 1$

- $\cos^{-1}(-x) = \pi - \cos^{-1} x \ \forall \ x \in [-1, 1]$

  $\cot^{-1}(-x) = \pi - \cot^{-1} x \ \forall \ x \in R$

  $\sec^{-1}(-x) = \pi - \sec^{-1} x \ \forall \ |x| \geq 1$

- $\sin^{-1} X + \cos^{-1} X = \frac{\pi}{2}, X \in [-1, 1]$

  $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \ \forall \ x \in R$

  $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2} \ \forall \ |x| \geq 1$

- $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy} & if\ xy < 1 \\ \pi + \tan^{-1} \frac{x+y}{1-xy} & if\ xy > 1 ; x > 0 \\ & y > 0 \\ -\pi + \tan^{-1} \frac{x+y}{1-xy} & if\ xy > 1 ; x < 0 \\ & y < 0 \end{cases}$

- $\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x-y}{1+xy} & if\ xy > -1 \\ \pi + \tan^{-1} \frac{x-y}{1+xy} & if\ xy < -1 ; x > 0 \\ & y < 0 \\ -\pi + \tan^{-1} \frac{x-y}{1+xy} & if\ xy < -1 ; x < 0 \\ & y > 0 \end{cases}$

- $2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2}\right), |x| < 1$

  $2\tan^{-1} x = \sin^{-1} \left(\frac{2x}{1-x^2}\right), |x| \leq 1,$

  $2\tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2}\right), x \geq 0.$

- $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

  $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Write the principal value of

(i) $\sin^{-1}\left(-\sqrt{3/2}\right)$ **(ii)** $\cos^{-1}\left(\sqrt{3}/2\right)$

(iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **(iv)** $\text{cosec}^{-1}(-2)$

(v) $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **(vi)** $\sec^{-1}(-2)$.

(vii) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{-1}{2}\right) + \tan^{-1}\left(-1/\sqrt{3}\right)$

2. What is the value of the following functions (using principal value)

(i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **(ii)** $\sin^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(iii) $\tan^{-1}(1) - \cot^{-1}(-1)$ **(iv)** $\text{cosec}^{-1}\left(\sqrt{2}\right) + \sec^{-1}\left(\sqrt{2}\right)$

(v) $\tan^{-1}(1) + \cot^{-1}(1) + \sin^{-1}(1)$. **(vi)** $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$

(vii) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$ **(viii)** $\text{cosec}^{-1}\left(\text{cosec}\frac{3\pi}{4}\right)$

(ix) $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right\}$

3. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, find $\cot^{-1}x + \cot^{-1}y$.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

4. Show that: $\tan^{-1}\left[\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-cosx}}\right] = \frac{\pi}{4}+\frac{x}{2}$; $x \in [0,\pi]$

5. Prove that :

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) - \cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right) = \frac{\pi}{4} \quad x \in (0, \pi/2).$$

6. Prove that $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a} = \cos^{-1}\left(\frac{\sqrt{a^2-x^2}}{a}\right)$.

7. prove that:

$$\cot^{-1}\left[2\, tan\left(cos^{-1}\frac{8}{17}\right)\right] + \tan^{-1}\left[2\tan\left(sin^{-1}\frac{8}{17}\right)\right] = \tan^{-1}\left(\frac{300}{161}\right)$$

8. Prove that:

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\, cos^{-1}\ x^2$$

9. Solve:

$$\cot^{-1} 2x + \cot^{-1} 3x = \frac{\pi}{4}$$

10. Prove that:

$$\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}, m, n > 0$$

11. Prove that:

$$\tan\left[\frac{\pi}{4}+\frac{1}{2}tan^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\left(\frac{a}{b}\right)\right] = \frac{2\sqrt{a^2+b^2}}{b}$$

12. Solve for x, $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{-2x}{1-x^2}\right) = \frac{2\pi}{3}$

13. Prove that: $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

14. Solve for x, $\tan(\cos^{-1} x) = \sin(\tan^{-1} 2)$; x > 0

15. If y = $\cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$, then prove that sin y = $tan^2\left(\frac{x}{2}\right)$

16. Evaluate: $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{\sqrt{11}}\right)\right]$

17. Prove that: $\tan^{-1}\left(\frac{a\cos x - b\ \sin x}{b\cos x + a\ \sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x$

18. Prove that:

$$\cot\left\{\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right\} + \cos^{-1}(1-2x^2) + \cos^{-1}(2x^2-1) = \pi, x > 0$$

19. Prove that:

$\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right) = 0$ where a, b, c > 0

20. If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, then

prove that a + b + c = abc

21. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$

[Hint: Let $\cos^{-1} x$ =A, $\cos^{-1} y = B, \cos^{-1} z = C$ then A + B + C = π or A +B = π-c Take cos on both the sides].

22. If $\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + \cdots + \tan^{-1}\left(\frac{1}{1+n.(n+1)}\right) = \tan^{-1}$ ø then find the value of ø.

23. If $(\tan^{-1} x^2) + (\cot^{-1} x^2) = \frac{5\pi^2}{8}$ then find x.

24. If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$, then find x.

25. Solve the following for x

(i) $\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3}\, x) = -\frac{\pi}{2}$

(ii) $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

(iii) $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

(iv) $\sin^{-1}\left(\frac{x}{2}\right) + \cos^{-1} x = \frac{\pi}{6}$

26. If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \alpha$, then prove that

$9x^2$-12xy $\cos\propto + 4y^2 = 36\sin^{-1}\propto$

27. Prove that: $\tan^{-1}\left[\frac{3\sin 2\emptyset}{5+3\cos 2\emptyset}\right] + \tan^{-1}\left[\frac{1}{4}\tan\emptyset\right] = \emptyset$

28. Prove that: $\cot^{-1}\left[\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{2}\right)\right] = \frac{\pi}{2}$

29. Prove that: $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) = \cos^{-1}\left(\frac{a\ \cos x + b}{a+b\ \cos x}\right)$

30. Prove that: $2\tan^{-1}[\tan\propto/2\ \tan\beta/2] = \cos^{-1}\left[\frac{\cos\propto + \cos\beta}{1+\cos\alpha\ \cos\beta}\right]$

ANSWERS

1. **(i)** $-\frac{\pi}{3}$ **(ii)** $\frac{\pi}{6}$ **(iii)** $\frac{-\pi}{6}$ **(iv)** $\frac{-\pi}{6}$

(v) $\frac{\pi}{3}$ **(vi)** $\frac{2\pi}{3}$ **(vii)** $\frac{\pi}{6}$

2. **(i)** 0 **(ii)** $\frac{-\pi}{3}$ **(iii)** $-\frac{\pi}{2}$

(iv) $\frac{\pi}{2}$ **(v)** π **(vi)** $\frac{\pi}{5}$

(vii) $\frac{-\pi}{6}$ **(Viii)** $\frac{\pi}{4}$ **(ix)** -1

3. $\pi/5$

9. 1

12. $\tan\frac{\pi}{12} = 2 - \sqrt{3}$

14. $\frac{\sqrt{5}}{3}$

16. $\frac{\sqrt{11}-3}{\sqrt{2}}$

20. **Hint:** Let $\tan^{-1} a = \alpha$

$\tan^{-1} b = \beta$

$\tan^{-1} c = Y$

Then given, $\alpha + \beta + \gamma = \pi$

$\alpha + \beta = \pi - Y$

Take tangent on both sides

$\tan(\alpha + \beta) = \tan(\pi - Y)$

22. $\emptyset = \frac{n}{n=2}$

23. X = -1

24. $x = -\frac{1}{2}$

25. **(i)** $\mathrm{x} = -\frac{1}{12}$ **(ii)** $\mathrm{x} = 0, \frac{1}{2}$

(iii) x=13 **(iv)** x=1

CHAPTER: 3 and 4

MATRICES And DETERMINANTS

IMPORTANT POINTS TO REMEMBER

Matrix: **It** is an ordered rectangular arrangement of numbers (or functions). The numbers (or functions) are called the **elements** of the matrix. Horizontal line of elements is **row** of matrix. Vertical line of elements is **column** of matrix.

Numbers written in the horizontal line form a row of the matrix. Number written in the vertical line form a column of the matrix.

Order of Matrix with '*m*' rows and '*n*' columns is $m \times n$ *(read as m by n).*

Types of Matrices

- A **row matrix** has only one row (order:$1 \times n$)
- A **column matrix** has only one column (order: $m \times 1$)
- A **square matrix** has number of rows equal to number of columns (order: $m \times m$ or $n \times n$.)
- A **diagonal matrix** is a square matrix with all non-diagonal elements equal to zero and diagonal elements not all zeroes.
- A **scalar matrix** is a diagonal matrix in which all diagonal elements are equal.
- An **identity matrix** is a scalar matrix in which each diagonal element is 1 (unity).
- **A zero matrix** or **null matrix** is the matrix having all elements zero.

- **Equal matrices:** two matrices A = $[a_{ij}]$ and $[\ b_{ij}]$ are equal if

  (a) Both have same order

  (b) a_{ij} = b_{ij} ∀ i and j

Operations on matrices

- Two matrices can be added or subtracted, if both have same order.
- If A= $[a_{ij}]_{m \times n}$ and B =$[b_{ij}]_{m \times n}$, then

  A $\pm$ B = $[a_{ij} \pm\ b_{ij}]_{m \times n}$

- $\lambda A = [\lambda\, a_{ij}]_{m \times n}$ where λ is a scalar
- Two matrices A and B can be multiplied if number of columns in A is equal to number of rows in B.

  If $A =\ [a_{ij}]_{m \times n}$ and $[b_{jk}]_{n \times p}$

  Then $AB = [c_{ik}]_{m \times p}$ where $c_{ik} =\ \sum_{j=i}^{n} a_{ij}\ b_{jk}$

Properties

- If A, B and C are matrices of same order, then

  (i) A+B = B+A

  (ii) (A+B)+C= A+(B+C)

  (iii) A+O = O+A=A

  (iv) A+(-A) = O

- If A, B and C are matrices and λ is any scalar, then

    (ii) AB ≠ BA

    (iii) (AB) C = A(BC)

    (iv) A(B+C) = AB+AC

    (v) AB=O need not necessarily imply A=O or B=O

    (vi) λ (AB)= (λA) B=A (λB)

    Transpose of a Matrix: Let A be any matrix. Interchange rows and columns of A. The new matrix so obtained is transpose of A donated by A'/A^T.

    [order of A = m × n ⇨ order of A' = n × m]

    Properties of transpose matrices A and B are:

    (i) $(A')' = A$

    (ii) $(kA)' = kA'$ (k= constant)

    (iii) $(A + B)' = A' + B'$

    (iv) $(AB)' = B'A'$

    Symmetric Matrix and Skew-Symmetric matrix

- A square matrix $A = [a_{ij}]$ is symmetric if $A' = A$ i.e. . $a_{ij} = a_{ji}$ ∀ i and j
- A square matrix A= $[a_{ij}]$ is skew-symmetric if $A' = -A$ i.e. $a_{ij} = - a_{ji}$ ∀ i and j (All diagonal elements are zero in skew-symmetric matrix)

Determinant: to every square matrix A= $[a_{ij}]$ of order n×n, we can associate a number (real or complex). This is called determinant of A (det A or $|A|$).

Properties of Determinants

I) $|AB| = |A|\,|B|$

II) $|A'| = |A|$

III) If we interchange rows (or columns), sign of $|A|$ changes.

IV) Value of $|A|$ is zero, if any two rows or columns in A are identical (or proportional).

V) $\begin{vmatrix} a+b & x \\ c+d & y \end{vmatrix} = \begin{vmatrix} a & x \\ c & y \end{vmatrix} + \begin{vmatrix} b & x \\ d & y \end{vmatrix}$

VI) $R_i \rightarrow R_i \pm aR_j$ or $C_i \rightarrow C_i \pm bC_j$ does not alter the value of $|A|$.

VII) $|k\,A|_{n\times n} = k^n |A|_{n\times n}$ (k= scalar)

VIII) $K\,|A|$ means multiplying only one row (or column) by k.

IX) Area of triangle with vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$is:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear if area of triangle is zero

Minors and Cofactors

- Minor (M_{ij}) of a_{ij} in A is the determinant obtained by deleting i^{th} row and j^{th} column.
- Cofactor of a_{ij} , $A_{ij} = (-1)^{i+j}\ M_{ij}$

Adjoint of a Square Matrix

adj A= transpose of the square matrix A whose elements have been replaced by their cofactors.

Properties of adj A: For any square matrix A of order n:

(i) $A(adj\ A) = (adj\ A)\ A = |A|\ \mathrm{I}$

(ii) $|adj\ A| = |A|^{n-1}$

(iii) $adj\ (AB) = (adj\ B)\ (adj\ A)$.

(iv) $|k\ adj\ A| = k^n\ |A|^{n-1}$.

Singular Matrix: A square matrix A is singular if $|A| = 0$.

Inverse of a Matrix: An inverse of a square matrix exists if and only if it is non-singular.

$\mathrm{A}^{-1} = \frac{1}{|A|}$ adj A

Properties of Inverse matrix

(i) $AA^{-1} = A^{-1}A = I$

(ii) $(A^{-1})^{-1} = A$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A')^{-1} = (A^{-1})'$

(v) $|A'| = \frac{1}{|A|}$, $|A| \neq 0$

Solution of system of equations using matrices:

If $AX = B$ is a matrix equation, then

AX=B $\Rightarrow A^{-1}AX = A^{-1}B \Rightarrow I\,X = A^{-1}B \Rightarrow X = A^{-1}B$ gives the solution.

Criterion of consistency of system of liner equations

(i) If $|A| \neq 0$, system is consistent and has a unique solution.

(ii) If $|A| = 0$ and $(adjA)\ B \neq 0$, then the system AX=B is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(adj\ A)\ B = 0$ then system is consistent and has infinitely many solutions.

VERY SHORT ANSWER TYPE QUESTIONS (1 Mark)

1. If $[1\ x\ 1]\begin{bmatrix}1 & 3 & 2\\0 & 5 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\1\\-2\end{bmatrix} = 0$, then What is the value of x?

2. For what value of λ, the matrix A is a singular matrix where

$$A = \begin{bmatrix}1 & 3 & \lambda+2\\2 & 4 & 8\\3 & 5 & 10\end{bmatrix}$$

3. Find the value of A^2, if

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\a & b & -1\end{bmatrix}$$

4. If $A = \begin{bmatrix}a & b\\b & a\end{bmatrix}$ and $A^2 = \begin{bmatrix}\alpha & \beta\\\beta & \alpha\end{bmatrix}$, then find the value of $\propto\ and\ \beta$.

5. If A is a square matrix such that $A^2 = I$, then write the value of $(A-I)^3 + (A+I)^3 - 7$ A in simplest form.

6. Write the value of Δ, if

$$\Delta = \begin{vmatrix}x+y & y+z & z+x\\z & x & y\\-3 & -3 & -3\end{vmatrix}$$

7. If $\begin{bmatrix}x-y & z\\2x-y & w\end{bmatrix} = \begin{bmatrix}-1 & 4\\0 & 5\end{bmatrix}$, find the value of x+y.

8. If A is a 3 × 3 matrix, $|A| \neq 0$ and $|3A| = K|A|$, then write the value of K.

9. If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is a symmetric matrix, then write the value of x.

10. Matrix $A = \begin{bmatrix} 0 & 2a & -2 \\ 3 & 1 & 3 \\ 3b & 3 & -1 \end{bmatrix}$ is given to be symmetric, find the value of a and b.

11. For any 2 × 2 matrix A, if A (adjoint A) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then find $|A|$.

12. Find X, if A + X = I, where

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 3 & 4 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

13. If $U = [2 \quad -3 \quad 4], V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$,X= $[0 \quad 2 \quad 3]$ and $Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then find UV+XY.

14. If $\begin{bmatrix} 2 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 16 & 15 \end{bmatrix}$

write the equation after applying elementary column transformation $C_2 \to C_2 + 2C_1$

15. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then find the value of A^3.

16. Find the value of $a_{23} + a_{32}$ in the matrix

$A = [a_{ij}]_{3\times 3}$ where $a_{ij} = \begin{cases} |2\mathrm{i} - \mathrm{j}| & \text{if i} > j \\ -\mathrm{i} + 2\mathrm{j} + 3 & \text{if i} < j \end{cases}$

17. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then find $|A^2|$.

18. For what value of x, is the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & x & -3 \\ 2 & 3 & 0 \end{bmatrix} \text{ a skew-symmetric matrix}$$

19. If $A = \begin{bmatrix} \sin 15^\circ & \cos 15^\circ \\ -\sin 75^\circ & \cos 75^\circ \end{bmatrix}$, then evaluate $|A|$.

20. If A is a square matrix, expressed as A= X+Y where X is symmetric and Y is skew-symmetric, then write the values of X and Y.

21. Write a matrix of order 3 × 3 which is both symmetric and skew-symmetric matrix.

22. What positive value of x makes the following pair of determinants equal?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \quad \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

23. $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, find the value of $5A_{31} + 3A_{32} + 8A_{33}$.

24. If $A = \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}$, find $|A\ (adjA)|$

25. Find the minimum value of. $2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + sin\theta & 1 \\ 1 & 1 & 1 + cos\theta \end{vmatrix}$

26. If A and B are square matrices of order 3 and $|A| = 5$ and $|B| = 3,$ then find the value of $|3AB|$.

27. Evaluate $\begin{vmatrix} 3 + 2i & -6i \\ 2i & 3 - 2i \end{vmatrix}$

28. Without expanding, find the value of $\begin{vmatrix} cosec^2\theta & cot^2\theta & 1 \\ cot^2\theta & cosec^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$

29. Using determinants, find the equation of line passing through (0,3) and (1,1).

30. If A be any square matrix of order 3 × 3 and $|A| = 5,$ then find the value of $|adj\ (adjA)|$

31. What is the number of all possible matrices of order 2 × 3 with each entry 0,1 or 2.

32. Given a square matrix A of order 3 × 3 such that $|A|$=12, find the value of $|A\ adj\ A|$

33. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ find $|(A^{-1})^{-1}|$

34. If $A = [-1 \quad 2 \quad 3]$ and $B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ find $|AB|$

35. Find $|A\ (adjoint\ A)|$ and $|adjoint\ A|$, if $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

36. If $\begin{vmatrix} a & y & z \\ x & b & z \\ x & y & c \end{vmatrix} = 0,$ then prove that $\frac{a}{a-x} + \frac{b}{b-y} + \frac{c}{c-z} = 2$

37. If a ≠ b ≠ c, find the value of x which satisfies the equation

$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$$

38. Using properties of determinants, show that

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 0$$

39. Find the value of $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

40. If $A = \begin{bmatrix} 5 & 3 \\ 12 & 7 \end{bmatrix}$, show that $A^2 - 12A - I = 0$. Hence find A^{-1}.

41. Find the matrix X so that $X \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 2 & 0 \end{bmatrix}$

42. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, verify that $A^2 - 4A - 5I = 0$.

43. Using elementary transformations find the inverse of the matrix

$A = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$

44. If $A = \begin{bmatrix} x & -2 \\ 3 & 7 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{7}{34} & \frac{1}{17} \\ -\frac{3}{34} & \frac{2}{17} \end{bmatrix}$, then find the value of x.

45. If $A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$, find B, such that $4A^{-1} + B = A^2$

46. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \propto \\ 1 & -2 & 3 \end{bmatrix}$ and B=A^{-1}, then find the value of $\propto$.

47. Find the value of X, such that $A^2 - 5A + 4I + X = 0$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

48. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A^1)^{-1}$

49. The monthly incomes of Mohan and Sohan are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves ₹ 15000/- per month, find their monthly incomes and expenditures using matrices.

50. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$, then verify that $(AB)^1 = B^1A^1$

51. If $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $and\ x^2 = -1$

Then show that $(A+B)^2 = A^2 + B^2$

52. Prove that $aI + bA + cA^2 = A^3$, if $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$

53. If $A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$, then find A^3.

54. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2 + 2AB,$ find *a* and *b.*

55. If $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$, then find the value of a, b and c. Such that $A^TA = I$

56. If $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} a^n & b(\frac{a^n - 1}{a - 1}) \\ 0 & 1 \end{bmatrix}$, for all *n* ∊ *N*.

57. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then find A^{-1} and hence prove that $A^2 - 4A - 5I = 0$.

58. Find the value of k, if: $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

59. If x, y and z ∈ R, and

$$\Delta=\begin{vmatrix} x & x+y & x+y+z \\ 2x & 5x+2y & 7x+5y+2z \\ 3x & 7x+3y & 9x+7y+3z \end{vmatrix}=-16\text{, then find value of x.}$$

60. Find the value of 'k' if $\begin{vmatrix} 1 & a^2 & a^4 \\ 1 & b^2 & b^4 \\ 1 & c^2 & c^4 \end{vmatrix}=k\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

Using properties of determinants, prove the following (Ques.No.-61 to 69)

61. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}=0$

62. $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ac & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}=-(a-b)(b-c)(c-a)(a^2+b^2+c^2)$

63. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}=3(a+b+c)(ab+bc+ca)$

64. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}=a^3+b^3+c^3-3abc$

65. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}=0$

66. $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}=(x+y+z)^3$

67. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

68. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc - a^3 - b^3 - c^3)$

69. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

70. Given $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$. Find the product AB and also find $(AB)^{-1}$

71. Using properties of determinants, solve for x:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

72. Evaluate: $\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}$

73. If $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0$ and $a \neq b \neq c$ then find the value of x.

74. Express the following matrix as the sum of symmetric and skew-symmetric matrices and verify your result.

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

75. If x = – 4 is a root of a Δ= $\begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0,$ then find the other two roots.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

76. Prove that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$ is divisible by (*x* + *y* + *z)* and hence find the quotient.

77. Using elementary transformations, find the inverse of the matrix

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

78. Using matrix method, solve the system of linear equations

$x - 2y = 10,\ 2x - y - z = 8$ and $-2y + z = 7$

79. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = \frac{A^2-3I}{2}$

80. Find the matrix x for which $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

81. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$, then show that $f(A) = 0$, using this result find A^5.

82. If $a + b + c = 0$ and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$, then show that either

$x = 0$ or $x = \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$

83. If $A + B + C = \pi$, then find the value of

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -sinB & 0 & tanA \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$

84. If $\Delta = \begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$ prove that Δ is negative.

85. Let three digit number A28, 339, 62C, where A,B,C are integers between 0 and 9, be divisible by fixed integer 'k', then show that determinant.

$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by 'k'

86. Using properties of determinants prove that:

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$$

87. Solve the following equation for x $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

88. Prove that: $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$

Where a, b, c are in A.P.

89. Prove that: $\begin{vmatrix} a & a+c & a-b \\ b-c & b & b+a \\ c+b & c-a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

90. If a, b, c are p^{th}, q^{th} and r^{th} terms respectively of a G.P. Prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

91. Without expanding prove that $\begin{vmatrix} o & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-p & o \end{vmatrix} = 0$

92. Prove that (x-2) (x-1) is factor of $\begin{vmatrix} 1 & 1 & x \\ \beta+1 & \beta+1 & \beta+x \\ 3 & x+1 & x+2 \end{vmatrix}$ and hence find the quotient.

93. Prove that:

$$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^{2)} & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc(a^2+b^2+c^2)^3$$

94. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

and use it to solve the system of equations:

$x - y + z = 4,\ x - 2y - 2z = 9,\ 2x + y + 3z = 1$

95. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, find A^{-1} and use it to solve the system of linear equations: $x + 2y + z = 4,\ -x + y + z = ,\ \ x - 3y + z = 2$

96. Solve given system of equations by matrix method:

$$3y + x = xy \ \ and \ \ y + x = 3xy$$

97. If $S_m = \alpha^m + \beta^m + \gamma^m$, then prove

$$\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = [(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)]^2$$

98. To raise money for an orphanage, students of three schools A, B and C organized an exhibition in their locality, where they sold paper bags, scrap books and pastel sheets made by them using recycled paper, at the rate of ₹ 20, ₹ 15 and ₹ 5 per unit respectively. School A sold 25 paper bags, 12 scrap books and 34 pastel sheets. School B sold 22 paper bags, 15 scrap books and 28 pastel sheets. While school C sold 26 paper bags, 18 scrap books and 36 pastel sheets. Using matrices, find the total amount raised by each school. By such exhibition, which values are inculcating in the students?

99. Two cricket teams honored their players for three values, excellent batting, to the point bowling and unparalleled fielding by giving ₹ x, ₹ y and ₹ z per player respectively. The first team paid respectively 2, 2 and 1 players for the above values with a total prize money of 11 lakhs, while the second team paid respectively 1,2 and 2 players for these values with a total prize money of ₹ 9 lakhs. If the total award money for one person each for these values amount to ₹ 6 lakhs, then express the above situation as a matrix equation and find award money per person for each value.

For which of the above mentioned values, would you like more and why?

ANSWERS

1. $\frac{1}{2}$

2. $\lambda = 4$

3. $A^2 = I_3$

4. $\alpha = a^2 + b^2,\ \beta = 2ab$

5. A

6. 0

7. 3

8. K=27

9. X = 5

10. a= $\frac{3}{2}$, $b = \frac{-2}{3}$

11. $|A| = 10$

12. $X = \begin{bmatrix} 0 & -4 & 1 \\ -3 & -3 & -7 \\ -5 & -1 & -5 \end{bmatrix}$

13. [20]

14. $\begin{bmatrix} 2 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} =$
$\begin{bmatrix} -4 & -17 \\ 16 & 47 \end{bmatrix}$

15. $A^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

16. 11

17. 0

18. x = 0

19. $|A| = 1$

20. $x = \frac{1}{2}(A + A'),\ y = \frac{1}{2}(A - A')$

21. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

22. x = 24

23. 0

24. 9

25. –1

26. 405

27. 1

28. 0

29. 3 – 2x

30. 625

31. 729

32. 1728

33. 11

34. –11

35. $a^9,\ a^6$

36. ....

37. x = 0

38. ....

39. $-5\sqrt{3}(5 - \sqrt{6})$

40. $A^{-1} = \begin{bmatrix} -7 & 3 \\ 12 & -5 \end{bmatrix}$

41. $\begin{bmatrix} 5 & 0 \\ -6/7 & 4/7 \end{bmatrix}$

42. .....

43. $\frac{1}{10}\begin{bmatrix} 7 & -1 \\ -4 & 2 \end{bmatrix}$

44. $x = 4$

45. $B = \begin{bmatrix} 2 & -15 \\ 0 & -3 \end{bmatrix}$

46. $\propto = 5$

47. $X = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 10 \\ 5 & -4 & -2 \end{bmatrix}$

48. $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

49. Incomes: Rs 90,000/- and Rs 1,20,000/-

Expenditures: Rs 75,000/- and Rs 10,5000/-

50. ....

51. ....

52. .....

53. $\begin{bmatrix} \cos 8\theta & \sin 8\theta \\ -\sin 8\theta & \cos 8\theta \end{bmatrix}$

54. A = -1; B = -2

55. $a = \pm\frac{1}{\sqrt{2}};\quad b = \pm\frac{1}{\sqrt{6}};$

$c = \pm\frac{1}{\sqrt{3}}$

56. ..

57. ..

58. K = 2

59. x =2

60. K =(a+b) (b+c)(c+a)

70. $AB = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$

72. **x = 4**

73. $4! = 24$

74. $x = \frac{-abc}{ab+bc+ca}$

75. $A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$

76. x = 1,3

77. $(x+y+z)(xy+yz+zx-x^2-y^2-z^2)^2$

78. $A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$

79. x = 0; y = -5; z = –3

80. $A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

81. $x = \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}$

82. $\begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$

84. 0

89. 0; 0; 3a

92. β

94. *Product* $= 8\,I$

x = 3, y = -2, z = -1

95. $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$ $x = \frac{9}{5}, y = \frac{2}{5}, z = \frac{7}{5}$

98. School A = ₹ 850

School B = ₹ 805

School C = ₹ 970

99. Excellent batting: 3 lakhs

point bowling: 2 lakhs

fielding: 1lakh

. CHAPTER 5

CONTINUITY AND DIFFERENTIATION

POINTS TO REMEMBER

- A function $f(x)$ is said to be continuous at $x = c$ iff $\lim_{x \to c} f(x) = f(c)$

  i.e., $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

- $f(x)$ is continuous in (a, b) iff it is continuous at $x = c$ $\forall$ $c \in (a, b)$.

- $f(x)$ is continuous in $[a, b]$ iff

  (i) f(x) is continuous in (a, b)

  (ii) $\lim_{x \to a^+} f(x) = f(a)$

  (iii) $\lim_{x \to b^-} f(x) = f(b)$

- Modulus functions is Continuous on R
- Trigonometric functions are continuous in their respective domains.
- Exponential function is continuous on R
- Every polynomial function is continuous on R.
- Greatest integer functions is continuous on all non-integral real numbers
- If $f(x)$ and $g(x)$ are two continuous functions at $x = a$ and $c \in R$ then

  (i) $f(x) \pm g(x)$ are also continuous functions at $x = a$.

  (ii) $g(x) \cdot f(x)$, $f(x) + c$, $cf(x)$, $|f(x)|$ are also continuous at $x = a$.

  (iii) $\frac{f(X)}{g(X)}$ is continuous at $x = a$ provided $g(a) \neq 0$.

- $f(x)$ is derivable at $x = c$ in its domain iff

  $\lim_{x \to c^-} \frac{f(X)-f(c)}{X-c} = \lim_{x \to c^+} \frac{f(X)-f(c)}{X-c}$, and is finite

  The value of above limit is denoted by $f'(c)$ and is called the derivative of $f(x)$ at $x = c$.

- $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$ (Product Rule)

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ (Quotient Rule)

- If y= f(u) and u=g(t) then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = f'(u)g'(t)$ (Chain Rule)

- If y =f(u), x=g(u) then,

  $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{f'(u)}{g'(u)}$$

- **Rolle's theorem:** If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) and $f(a) = f(b)$ then there exists at least one real number $c \in (a, b)$ such that $f'(c) = 0$.

- **Mean Value Theorem:** If $f(x)$ is continuous in $[a, b]$ and derivable in (a, b) then there exists at least one real number $c \in (a, b)$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. Let f(x) = sin x Cos x. write down the set of points of discontinuity of f (x).

2. Given f (x) = $\frac{1}{x+2}$, write down the set of points of discontinuity of f (f(x)).

3. For what value(s) of n, the function f(x) = $\begin{cases} x^n sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Is continuous at x = 0.

4. Write the set of points of continuity of

$f(x) = |x-1| + |x+1|$

5. Write the number of points of discontinuity of $f(x) = [x]$ in [3, 7].

6. Differentiate Sin (x^2) w.r.t. x^3 .

7. If $y = e^{\log(x^5)}$, find $\frac{dy}{dx}$.

8. If $f(x) = x^2 g(x)\ and\ g(1) = 6,\ g'(x) = 3, find\ the\ value\ of\ f'(1).$

9. If $y = x^x$, then find $\frac{dy}{dx}$

10. If $y = a\sin t, x = a\cos t$ then find $\frac{dy}{dx}$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

11. Examine the continuity of the following functions at the indicated points.

(I) $f(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$

(II) $f(x) = \begin{cases} x - [x] & x \neq 0 \\ 0 & x = 1 \end{cases}$ at $x = 1$

(III) $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$

(IV) $f(x) = \begin{cases} \frac{x-\cos(sin^{-1}x)}{1-\tan(sin^{-1}x)} & x \neq \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & x = \frac{1}{\sqrt{2}} \end{cases}$ at $x = \frac{1}{\sqrt{2}}$

12. For what values of constant K, the following functions are continuous at the indicated points.

(I) $f(x) = \begin{cases} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & x < 0 \\ \frac{2x+1}{x-1} & x > 0 \end{cases}$ at $x = 0$

(II) $f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)} & x \neq 0 \\ K & x = 0 \end{cases}$ at $x = 0$

(III) $f(x) = \begin{cases} \frac{1-cos4x}{x^2} & x < 0 \\ K & x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & x > 0 \end{cases}$ at $x = 0$

13. For what values a and b

$$f(x) = \begin{cases} \frac{x+2}{|x+2|} + a & if\ x < -2 \\ a + b & if\ x = -2 \\ \frac{x+2}{|x+2|} + 2b & if x > -2 \end{cases}$$

Is continuous at $x = -2$

14. Find the values of a, b and c for which the function

$$f(x) = \begin{cases} \frac{\sin[(a+1)x]+sinx}{x} & x < 0 \\ C & x = 0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$$

Is continuous at $x = 0$

15. $f(x) = \begin{cases} [x] + [-x] & x \neq 0 \\ \lambda & x = 0 \end{cases}$

Find the value of λ, *f* is continuous at $x = 0$?

16. Let $f(x) = \begin{cases} \frac{1-sin^3 x}{3cos^2 x} & ; \quad x < \frac{\pi}{2} \\ a & ; \quad x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & ; \quad x > \frac{\pi}{2} \end{cases}$

If $f(x)$ is continuous at $x = \frac{\pi}{2}$, find *a* and *b.*

17. If $f(x) = \begin{cases} x^3 + 3x + a & x \le 1 \\ bx + 2 & x > 1 \end{cases}$

Is everywhere differentiable, find the value of *a* and *b*.

18. For what value of p

$f(x) = \begin{cases} x^p \sin(1/\mathrm{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is derivable at $x = 0$

19. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ where $x \neq 0.$

20. If $y = x^{x^x}$, then find $\frac{dy}{dx}$.

21. Differentiate $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$ w.r.t. x.

22. If $(x+y)^{m+n} = x^m . y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

23. If $(x-y). e^{\frac{x}{x-y}} = a$, prove that $y = \frac{dy}{dx} + x = 2y$

24. If $x = \tan\left(\frac{1}{a}\log y\right)$ then show that

$(1+x^2)\frac{d2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$

25. If $y = x\log\left(\frac{x}{a+bx}\right)$ prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

26. Differentiate $\sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$ w.r.t x.

27. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$, prove that

$$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

28. If $f(x) = \sqrt{x^2+1},\ g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x - 3$ find $f^1\left[h^1\left(g^1(x)\right)\right]$.

29. If $\sec\theta - \cos\theta$ and $y = sec^n\theta - cos^n\theta$, then prove that $\frac{dy}{dx} = n\sqrt{\frac{y^2+4}{x^2+4}}$

30. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

31. If $x = a\ cos^3\theta, y = a\ sin^3\theta$ then find $\frac{d^2y}{dx^2}$

32. If $y = \tan^{-1}\left[\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right]$ where $0 < x < \frac{\pi}{2}$ find $\frac{dy}{dx}$

33. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

34. Verify Rolle's theorem for the function

$f(x) = e^x \sin 2x \qquad \left[0, \frac{\pi}{2}\right]$

35. Verify mean value theorem for the function

$f(x) = \sqrt{x^2 - 4} \qquad [2,4]$

36. If the Rolle's theorem holds for the function

$f(x) = x^3 + bx^2 + ax + 5$ on [1,3] with $c = \left(2 + \frac{1}{\sqrt{3}}\right)$

Fin the value of *a* and *b.*

37. If $y = \left[x + \sqrt{x^2 + 1}\right]^m$, show that $(x^2 + 1)y_2 + xy_1 - m^2y = 0.$

38. Differentiate $\sin^{-1}\left[\frac{3x+4\sqrt{1-x^2}}{5}\right]$ w.r.t x.

39. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

40. If $f: [-5,5] \to R$ is a differentiable function and $f'(x)$ does not vanish anywhere, then prove that $fr(-5) \neq f(5)$.

ANSWERS

1. { }

2. R

3. $\left\{-2, \frac{-5}{2}\right\}$

4. R

5. Points of discontinuity of $f(x)$ are 4,5,6,7

Note- At $x = 3, f(x) = [x]$ is continuous because $\lim_{x \to 3^+} f(x) = 3 = f(3)$

6. $\frac{2\cos(x^2)}{3x}$

7. $5x^4$

8. 15

9. $x^x(1 + \log x)$

10. $-\cot t$

11. (I) Continuous

(II) Discontinuous

(III) Not Continuous at $x = 0$

(IV) Continuous

12. (I) K = -1

(II) K = $1/2$

(III) K = 8

13. $a = 0,\ \ b = -1$

14. $a = \frac{-3}{2},\ b = R - \{0\},\ c = \frac{1}{2}$

15. $\lambda = -1$

16. $a = \frac{1}{2},\ b = 4$

17. $a = 3,\ b = 5$

18. P > 1

19. $-\frac{1}{2}$

20. $x^x \, x^{x^x} \left\{(1 + \log x) \log x + \frac{1}{x}\right\}$

21. $(x \cos x)^x [1 - x \; \tan x + (\log x \;\; \cos x)] + (x \sin x)^{1/x} \left[\frac{1 + x \; \cot x - \log(x \; \sin)^x}{x^2}\right]$

26. $\left[\frac{2^{x+1} 3^x}{1 + (36)^x}\right] \log 6$ [Hint: $\tan \theta = 6^x$]

28. $\frac{2}{\sqrt{5}}$

30. $\frac{dy}{dx} = \frac{-x^x (1 + \log x) - y \, x^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$

31. $\frac{d^2 y}{dx^2} = \frac{1}{3a} cosec\theta \; sec^4\theta$

32. $-1/2$

36. $a = 11, \; b = -6$

38. $\frac{1}{\sqrt{1 - x^2}}$

CHAPTER 6

APPLICATION OF DERIVATIVES

IMPORTANT POINTS TO REMEMBER

- **Rate of change:** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\frac{dy}{dx}]x = x_1$ or $f'(x_1)$represents the rate of change of y w.r.t. x at $x = x_1$.

- **Increasing and Decreasing Function**

  Let f be a real-valued function and let I be any interval in the domain of f. Then f is said to be

  a) Strictly increasing on I, if for all $x_{1,}$ $x_2 \in I$, we have

  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

  b) Increasing on I, if for all $x_{1,}$ $x_2 \in I$, we have

  $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

  c) Strictly decreasing in I, if for all $x_{1,}$ $x_2 \in I$, we have

  $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

  d) Decreasing on I, if for all $x_{1,}$ $x_2 \in I$, we have

  $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

- **Derivative Test:** Let f be a continuous function on [a, b] and differentiable on (a, b). Then

  a) f is strictly increasing on [a, b] if $f'(x) > 0$ for each $x \in (a, b)$.

b) f is increasing on [a, b] if $f'(x) \geq 0$ for each $x \in (a, b)$.

c) f is strictly decreasing on [a, b] if $f'(x) < 0$ for each $x \in (a, b)$.

d) f is decreasing on [a, b] if $f'(x) \leq 0$ for each $x \in (a, b)$.

e) f is constant function on [a, b] if $f'(x) = 0$ for each $x \in (a, b)$.

- **Tangents and Normals**

a) Equation of the tangent to the curve $y = f(x)$ at (x_1, y_1) is

$$\mathrm{y} - \mathrm{y}_1 = \left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{(\mathrm{x}_1, \mathrm{y}_1)} \quad (\mathrm{x} - \mathrm{x}_1)$$

b) Equation of the normal to thte curve $y = f(x)$ at (x_1, y_1) is

$$\mathrm{y} - \mathrm{y}_1 = \frac{-1}{\left[\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{(\mathrm{x}_1, \mathrm{y}_1)}} \quad (\mathrm{x} - \mathrm{x}_1)$$

- **Maxima and Minima**

a) Let f be a function and c be a point in the domain of f such that either f ‘(x)=0 or f ‘(x) does not exist are called critical points.

b) **First Derivative Test:** Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in I. Then

i. *f ´*(*x*) changes sign from positive to negative as x increases through *c*, then *c* is called the point of the local maxima.

ii. *f ´(x)* changes sign from negative to positive as *x* increases through *c*, then *c* is a point of *local minima*.

iii. *f ´*(*x*) does not change sign as *x* increases through *c*, then *c* is neither a point of *local maxima* nor a point of *local minima*.Such a point is called a point of *inflexion.*

c) **Second Derivative Test :** Let *f* be a function defined on an interval I and let $c \in$ I. Let f be twice differentiable at c. Then

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of *f*.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value $f(c)$ is local minimum value of *f*.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

Very Short Answer Type Questions (1 Mark)

1. Find the angle θ, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

2. Find the slope of the normal to the curve $x = a\cos^3\theta$ and $y = a\sin^3\theta \; at \; \theta = \frac{\pi}{4}$

3. A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with respect to its radius when the radius is 7cm.

4. Write the interval for which the function $f(x) = cos\, x, 0 \leq x \leq 2\pi$ is decreasing

5. If the rate of change of Area of a circle is equal to the rate of change of its diameter. Find the radius of the circle.

6. For what values of x is the rate of increasing of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

7. Find the point on the curve $y = x^2 - 2x + 3$ where the tangent is parallel to x-axis.

8. Find the maximum value of $f(x) = 2x^3 - 24x + 107$ in the interval [1,3]

9. Write the maximum value of $f(x) = \frac{\log x}{x}$,if it exists.

10. Find the least value of $f(x) = ax + \frac{b}{x}$, where a>0, b>0 and x>0.

11. The sum of the two number is 8, what will the maximum value of the sum of their reciprocals.

12. Find the interval in which the function $f(x) = x - e^x + \tan(\frac{2\pi}{7})$increases.

13. For the curve $y = (2x + 1)^3$ find the rate of change of slope of the tangent.

14. Find the Co-ordinates of the point on the curve $y^2 = 3 - 4x$, where tangent is parallel to the line$2x + y - 2 = 0$.

15. Find the value of a for which the function $f(x) = x^2 - 2ax + 6, x > 0$ is strictly increasing.

Rate of Change (4 Mark Questions)

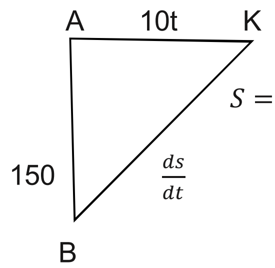
16. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm. Why is child labour not good for society?

17. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $\frac{3}{2}$c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

18. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.

19. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.

HINT: →



$S = \sqrt{150^2 + (10t^2)}$

Speed = $\frac{Dis}{t}$ $\quad 10 = \frac{Dis}{t} = AK \quad \left.\frac{ds}{dt}\right\} t = 20$

20. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?

21. The sides of an equilateral triangle are increasing at the rate of 2cm/s. Find the rate at which the area increases, when the side is 10cm.

22. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.

23. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is $\tan^{-1}(0.5)$. water is poured into it at a constant rate of$5m^3/h$. Find the rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

24. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

25. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of $1.5m^3/min$. find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

26. $x \ and \ y$ are the sides of two squares such that$y = x - x^2$. Find the rate of change of area of the second square w.r.t. the area of the first square.

27. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

28. If the areas of a circle increases at a uniform rate, then prove that the perimeter various inversely as the radius.

Increasing and Decreasing

29. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all $x \in R$. Find its value when the rate of increase of f(x) is least.

    [Hint: Rate of increase is least when $f'(x)$ is least.]

30. Determine whether the following function is increasing or decreasing in the given interval: $f(x) = \cos\left(2x + \frac{\pi}{4}\right), \frac{3\pi}{8} \leq x \leq \frac{5\pi}{8}$.

31. Determine for which values of x, the function y=$x^4 - \frac{4x^3}{3}$ is increasing and for which it is decreasing.

32. Find the interval of increasing and decreasing of the function $f(x) = \frac{\log x}{x}$

33. Find the interval of increasing and decreasing of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$.

34. Show that $f(x) = x^2e^{-x}, \quad 0 \leq x \leq 2$ is increasing in the indicated interval.

35. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0_1 \frac{\pi}{2}\right]$.

36. Find the intervals in which the following function is decreasing.

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

37. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

38. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$,is strictly increasing the interval $\left(0, \frac{\pi}{4}\right)$.

39. Find the interval in which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is increasing or decreasing.

40. Find the interval in which the function given by

$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$

(1) strictly increasing

(2) strictly decreasing

Tangent and Normal

41. Find the equation of the tangent to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

42. Find the equation of the tangent to the curve $y = x^2 - 2x + 7$ which is

(1) Parallel to the line $2x - y + 9 = 0$

(2) Perpendicular to the line $5y - 15x = 13$

43. Find the co-ordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

44. Find a point on the parabola $f(x) = (x - 3)^2$ where the tangent is parallel to the chord joining the points (3,0) and (4,1).

45. Find the equation of the normal to the curve $y = e^{2x} + x^2\ at\ x = 0$. also find the distance from origin to the line.

46. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point, where the curve intersects the axis of y.

47. A what point on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to

(1) X – axis

(2) Y – axis

48. Show that the equation of the normal at any point 'θ' on the curve $x = 3\cos\theta - cos^3\theta$, $y = 3\sin\theta - sin^3\theta$ is
4$(y\ cos^3\theta - xsin^3\theta)$=3 sin 4 θ.

49. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

50. For the curve $y = 5x - 2x^{3,}$ if x increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when x=3?

51. Find the condition for the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ to interest orthogonally.

52. Show that the curves $y = a^x$ and $y = b^x$, a > b > 0 intersect at an angle of $\tan^{-1}\left(\left|\frac{\log\left|\frac{a}{b}\right|}{1+\log a\ \log b}\right|\right)$

53. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

54. Find the equation of the normal at a point on the curve $x^2 = 4y$, which passes through the point (1, 2). Also find the equation of the corresponding tangent.

55. Find the point on the curve $9y^2 = x^3$ where the normal to the curve makes equal intercepts with the axes.

56. Show that the tangents to the curve $y = 2x^3 - 3$ at the point where x = 2 and x = -2 are parallel.

APPROXIMATION

Use differentials to find the approximate value of (Ques.57 to 62)

57. $(66)^{1/3}$

58. $\sqrt{401}$

59. $\sqrt{0.37}$

60. $\sqrt{25.3}$

61. $(3.968)^{3/2}$

62. $(26.57)^{1/3}$

63. Find the value of $log_{10}(10.1)$ given that $log_{10}\mathrm{e} = 0.4343$.

64. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

65. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.

66. Find the approximate value of f(2.01) where $f(x) = x^3 - 4x + 7$.

67. Find the approximate value of $\frac{1}{\sqrt{25.1}}$, using differentials.

68. The radius of a sphere shrinks from 10 cm. to 9.8 cm. Find the approximately decrease in its volume.

Maxima and Minima (4 Mark Questions)

69. Find the maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

70. Find the absolute maximum value and absolute minimum value of the following question $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$ in [-2, 2.5]

71. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval [0, 1]

72. Find the absolute maximum and absolute minimum value of $f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

73. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Maxima and Minima (6 Mark Question)

74. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

75. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

76. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

77. The sum of the surface areas of cuboids with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x = 3 radius of the sphere. Also find the minimum value of the sum of their volumes.

78. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

79. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

80. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

81. Show that the volume of the greatest cylinder which can be inscribed in a cone of height H and semi-vertical angle $\alpha\ is\ \frac{4}{27}\ \pi h^3 tan^2 \alpha.$ Also show that height of the cylinder is $\frac{h}{3}$

82. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2,1).

83. Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$.

84. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?

85. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

86. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Answers

1. $\frac{\pi}{3}$
2. 1
3. $196\pi\, cm^2$
4. $[0, \pi]$
5. $\frac{1}{\pi}$ *units*
6. $3, \frac{1}{3}$
7. (1,2)
8. 89
9. $\frac{1}{e}$
10. $2\sqrt{ab}$
11. $\frac{1}{2}$
12. $(-\infty, 0)$
13. 0
14. $\left(\frac{1}{2}, 1\right)$
15. $a \leq 0$
16. $\frac{1}{\pi} cm/s$
17. $\frac{3}{8\pi}$ cm / min
18. …….
19. 8 m/sec.
20. 3000 L/s
21. $10\sqrt{3}$
22. 3 km/h
23. $\frac{35}{88}$ m/h
24. ….
25. $\frac{6}{49\pi}$ m/min.
26. $1 - 3x + 2x^2$
27. 72

28. ...

29. 25

30. Increasing

31. Increasing for all $x \geq 1$

Decreasing for all $x \leq 1$

32. Increasing on [o, e]

Decreasing on [e, ∞)

33. Increasing on

$[0, \frac{3\pi}{4}]$ U $[\frac{7\pi}{4}, 2\pi]$

Decreasing on $[\frac{3\pi}{4} . \frac{7\pi}{4}]$

36. $(-\infty, 1]\text{U}[2, 3]$

37. [1,∞]

39. increasing on [0, ∞)

Decreasing (-∞, 0]

40. Strictly increasing

[-2,1] U [3, ∞)

Strictly decreasing

(-∞, -2] U [1, 3]

41. $\sqrt{2}bx - ay - ab = 0$

42. (1) $y - 2x - 3 = 0$

(2) $36y + 12x - 227 = 0$

43. (4, 4)

44. $\left(\frac{7}{2}, \frac{1}{4}\right)$

45. $2y + x - 2 = 0, \frac{2}{\sqrt{5}}$

47. (1) (1, 0) and (1, 4)

(2) (3, 2) and (-1, 2)

50. decrease 72 units/sec.

51. $a^2 = b^2$

53. $2x + 3my - 3a\,m^4 - 2am^2 = 0$

54. $x + y = 3, \quad y = x - 1$

55. $\left(4, \pm\frac{8}{3}\right)$

57. 4.042

58. 20.025

59. 0.1924

60. 5.03

61. 7.904

62. 2.984

63. 1.004343

64. $\pi\ cm^2$

65. 0.3%

66. 7.08

67. 0.198

68. $80\pi cm^3$

69. max. value =$\frac{3}{4}$, mim value =$\frac{1}{2}$

70. ab. Max.= $\frac{157}{8}$, ab. Min. = $\frac{-7}{4}$

71. max. value = 0,

min. value = $\frac{-3}{5}\ \left[\frac{2}{5}\right]^{2/3}$

72. ab. Max = 14 at x = 9

ab. Min.= $\frac{-3}{4^{4/3}}$ at $x = \frac{5}{4}$

73. π

77. $18r^3 + (36)(27)\pi r^3$

82. (1, 2)

83. $\frac{3\sqrt{2}}{8}$

84. $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

86. 2ab sq. Units.

CHAPTER 7

INTEGRALS

POINTS TO REMEMBER

- Integration or anti derivative is the reverse process of Differentiation.
- Let $\frac{d}{dx}F(x) = f(x)$ then we write $\int f(x)\,dx = F(x) + c$.
- These integrals are called indefinite integrals and c is called constant of integration.
- From geometrical point of view an indefinite integral is collection of family of curves each of which is obtained by translating one of the curves parallel to itself upwards or downwards along y-axis.

STANDARD FORMULAE

1. $$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log_e|x| + c & n = -1 \end{cases}$$

2. $$\int (ax+b)^n\,dx = \begin{cases} \frac{(ax+b)^{n+1}}{(n+1)a} + c & n \neq -1 \\ \frac{1}{a}\log|ax+b| + c & n = -1 \end{cases}$$

3. $\int \sin x\,dx = -\cos x + c.$

4. $\int \cos x\,dx = \sin x + c$

5. $\int \tan x.\,dx = -\log|\cos x| + c = \log|\sec x| + c.$

6. $\int \cot x\,dx = \log|\sin x| + c.$

7. $\int \sec^2 x\,dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

11. $\int \sec x \, dx = \log|\sec x + \tan x| + c$

$$= \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$$

12. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

$$= \log\left|\tan\frac{x}{2}\right| + c$$

13. $\int e^x \, dx = e^x + c$

14. $\int a^x \, dx = \frac{a^x}{\log a} + c$

15. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c, |x| < 1$

$$= -\cos^{-1} x + c$$

16. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$$= -\cot^{-1} x + c$$

17. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + c, |x| > 1$

$$= -\operatorname{cosec}^{-1} x + c$$

18. $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

19. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

20. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

21. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$

22. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x + \sqrt{a^2 + x^2}\right| + c$

23. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

24. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

25. $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$

26. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

RULES OF INTEGRATION

1. $\int (f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_x(x)dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \ldots\ldots \pm \int f_x(x)dx$

2. $\int k.f(x)dx = k\int f(x)dx.$

3. $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$

INTEGRATION BY SUBSTITUTION

1. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

2. $\int [f(x)]^n\ f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$

3. $\int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{-n+1}}{-n+1} + c$

INTEGRATION BY PARTS

$$\int f(x)\, g(x)\, dx = f(x)\left[\int g(x)dx\right] - \int f'(x) \left[\int g(x)dx\right] dx$$

DEFINITE INTEGRALS

$$\int_a^b f(x)dx = F(b) - F(a), \text{where } F(x) = \int f(x)\ dx$$

DEFINITE INTEGRAL AS A LIMIT OF SUMS.

$$\int_a^b f(x)dx = \lim_{h\to o} h\,[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+n-1h)]$$

$$Where\ h = \frac{b-a}{h} \quad or \int_a^b f(x)dx = \lim_{h\to o}[h \textstyle\sum_{r=1}^{n} f(a+rh)]$$

PROPERTIES OF DEFINITE INTEGRAL

1. $\int_a^b f(x) = -\int_b^a f(x)dx$

2. $\int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$

3. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$

4. (*i*) $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx.$

   (ii) $\int_o^a f(x)\ dx = \int_o^a f(a-x)dx$

5. $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \qquad if\ f\ (x)\ is\ even\ function$

6. $\int_{-a}^{a} f\ (x)dx = 0 \int_{0}^{a} \quad if\ f\ (x)\ is\ an\ odd\ function$

7. $\int_{0}^{2a} f\ (x)dx = \begin{cases} 2\int_{0}^{a} f\ (x)dx, & \text{if } f\,(2a-x) = f(x) \\ 0 & \text{if } f\,(2a-x) = -f\,(x) \end{cases}$

Very Short Answer Type Questions (1 Mark)

Evaluate the following integrals:

1. $\int \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}\right)\ dx$

2. $\int_{-1}^{1} e^{|x|}\ dx$

3. $\int \frac{dx}{1-\sin^2 x}$

4. $\int \left(\frac{x}{a} + \frac{a}{x} + x^a + a^x + ax\right)\ dx$

5. $\int_{-1}^{1} x^{99} \cos^4 x\ dx$

6. $\int \frac{1}{x\ \log x \log(\log x)} dx$

7. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \log\left(\frac{1+x}{1-x}\right) dx$

8. $\int \left(e^{a \log x} + e^{x \log a}\right) dx$

9. $\int \left(\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\right)\ dx$

10. $\int_{-\pi/2}^{\pi/2} \sin^7 x\ dx$

11. $\int \sqrt{10 - 4x + x^2}\ dx$

12. $\int_{-1}^{1} x^3|x|\, dx$

13. $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

14. $\int_{-2}^{2} \frac{dx}{1+|x-1|}$

15. $\int e^{-\log x}\, dx$

16. $\int \frac{e^x}{a^x}\, dx$

17. $\int e^x 2^x\, dx$

18. $\int \frac{x}{\sqrt{x+1}}\, dx$

19. $\int \frac{x}{(x+1)^2}\, dx$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

21. $\int \cos^2 \alpha\, dx$

22. $\int \frac{1}{x\cos\alpha+1}\, dx$

23. $\int \sec x \log(\sec x + \tan x)\, dx$

24. $\int \frac{1}{\cos\alpha + x\sin\alpha} dx$

25. $\int \frac{\sec^2(\log x)}{x}\, dx$

26. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\, dx$

27. $\int \frac{1}{x(2+3\log x)}\, dx$

28. $\int \frac{1-\sin x}{x+\cos x}\, dx$

29. $\int \frac{1-\cos x}{\sin x}\, dx$

30. $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}\,dx$

31. $\int \frac{(x+1)}{x}\,(x+\log x)\,dx$

32. $\int \left(\sqrt{ax}-\frac{1}{\sqrt{ax}}\right)^2 dx$

33. $\int_0^{\pi} |\cos x|\,dx$

34. $\int_0^2 [x]\,dx$ *where* $[x]$*is greatest integers function.*

35. $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

36. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$

37. $\int_{-2}^{1} \frac{|x|}{x}\,dx$

38. $\int_{-1}^{1} x\,|x|\,dx$

39. $\int x\sqrt{x+2}\,dx$

40. $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx$

41. $\int e^{\log(x+1)-\log x}\,dx$

42. $\int \frac{\sin x}{\sin 2x}\,dx$

43. $\int \sin x \sin 2x\,dx$

44. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\,dx$

45. $\int \frac{1}{\sqrt{x+1}+\sqrt{x+2}}\,dx$

46. $\int \frac{1}{\sec x+\tan x}\,dx$

47. $\int \frac{\sin^2 x}{1+\cos x}\,dx$

48. $\int \frac{1-\tan x}{1+\tan x}\, dx$

49. $\int \frac{a^x+b^x}{c^x}\, dx$

50. $\int_0^{\pi/2} \log\left[\frac{5+3\cos x}{5+3\sin x}\right] dx$

Short Answer Type Questions (4 Marks)

51. (I) $\int \frac{x\, cosec\,(\tan^{-1} x^2)}{1+x^4}\, dx$

(II) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\, dx$

(III) $\int \frac{1}{\sin(x-a)\ \sin(x-b)}\, dx$

(IV) $\int \frac{\cos\,(x+a)}{\cos\,(x-a)}\, dx$

(V) $\int \cos 2x\ \cos 4x\ \cos 6x\ dx$

(VI) $\int \tan 2x\ \tan 3x\ \tan 5x\, dx$

(VII) $\int \sin^2 x \cos^4 x\ dx$

(VIII) $\int \cot^3 x\, cosec^4 x\, dx$

(IX) $\int \frac{\sin x \cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}} dx$ [Hint: Put $a^2\sin^2 x + b^2\cos^2 x = t$ or t^2

(X) $\int \frac{1}{\sqrt{\cos^3 x\ \cos(x+a)}}\, dx$

(XI) $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x \cos^2 x}\, dx$

(XII) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

52. (i) $\int \frac{x}{x^4+x^2+1} dx$

(ii) $\int \frac{1}{x[6(\log x)^2+7\log x+2]}\, dx$

(iii) $\int \frac{1}{\sqrt{\sin^3 x \ \cos^5 x}}\, dx$

(iv) $\int \frac{x^2+1}{x^4+1} dx$

(v) $\int \frac{1}{\sqrt{(x-a)\,(x-b)}} dx$

(vi) $\int \frac{5x-2}{3x^2+2x+1}\, dx$

(vii) $\int \frac{x^2}{x^2+6x+12}\, dx$

(viii) $\int \frac{x+2}{\sqrt{4x-x^2}} dx$

(ix) $\int x\sqrt{1+x-x^2}\, dx$

(x) $\int \frac{\sin^4 x}{\cos^8 x}\, dx$

(xi) $\int \sqrt{\sec x - 1}\, dx$ [Hint: Multiply and divided by $\sqrt{\sec x + 1}$]

53. (I) $\int \frac{dx}{x(x^7+1)}$

(II) $\int \frac{3x+5}{x^3-x^2-x+1}\, dx$

(III) $\int \frac{\sin\theta\ \cos\theta}{\cos^2\theta - \cos\theta - 2}\ dx$

(iv) $\int \frac{dx}{(2-x)(x^2+3)}$

(v) $\int \frac{x^2+x+2}{(x-2)(x-1)}\ dx$

(vi) $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$

(vii) $\int \frac{dx}{(2x+1)(x^2+4)}$

(viii) $\int \frac{x^2-1}{x^4+x^2+1} dx$

(ix) $\int \sqrt{\tan x}\, dx$

(x) $\int \frac{dx}{\sin x - \sin 2x}$

54. Evaluate:

(I) $\int x^5\ \sin x^3\ dx$

(II) $\int \sec^3 x\ dx$

(III) $\int e^{ax}\ \cos(bx + c)\ dx$

(IV) $\int \sin^{-1}\left(\frac{6x}{1+9x^2}\right)\ dx$ [Hint: Put $3x = \tan\theta$]

(V) $\int \cos\sqrt{x}\, dx$

(VI) $\int x^3\ \tan^{-1} x\, dx$

(VII) $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right) dx$

(VIII) $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

(IX) $\int \sqrt{2ax - x^2}\, dx$

(X) $\int e^x \frac{(x^2+1)}{(x+1)^2} dx$

(XI) $\int x^3 \sin^{-1}\left(\frac{1}{x}\right) dx$

(XII) $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\} dx$ [Hint: Put $\begin{matrix}\log x = t \\ x = e^t\end{matrix}$]

(XIII) $\int (6x+5)\sqrt{6+x-x^2}\, dx$

(XIV) $\int \frac{1}{x^3+1} dx$

(XV) $\int \tan^{-1}\left(\frac{x-5}{1+5x}\right) dx$

(XVI) $\int \frac{dx}{5+4\cos x}$

55. Evaluate the following definite integrals:

(i) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\, \sin 2x}\, dx$

(ii) $\int_0^{\pi/2} \cos 2x\, \log \sin x\, dx$

(iii) $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx$

(iv) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

(v) $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\, dx$

(vi) $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right) dx$

(vii) $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\, dx$

(viii) $\int_0^1 x \log\left(1+\frac{x}{2}\right) dx$

(ix) $\int_{-1}^{1/2} |x \cos \pi x|\, dx$

(x) $\int_{-\pi}^{\pi} (\cos a\,x - \sin b\,x)^2\, dx$

56. Evaluate:

(i) $\int_2^5 [|x-2| + |x-3| + |x-4|]\, dx$

(ii) $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

(iii) $\int_{-1}^{1} e^{\tan^{-1} x} \left[\frac{1+x+x^2}{1+x^2}\right] dx$

(iv) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

(v) $\int_0^2 [x^2]\, dx$

(vi) $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x+\cos^4 x}\, dx$

(vii) $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ [*Hint: use* $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

57. Evaluate the following integrals:

(i) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

(ii) $\int_{-\pi/2}^{\pi/2} [\sin|x| + \cos|x|]\, dx$

(iii) $\int_{0}^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$

(iv) $\int_{0}^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx$

(v) $\int_{-a}^{a} \sqrt{\frac{a - x}{a + x}}\, dx$

58. Evaluate

(*i*) $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx \quad ; x = [0, 1]$

(*ii*) $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, dx$

(iii) $\int \frac{x^2\, e^2}{(x + 2)^2}\, dx$

(iv) $\int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$

(v) $\int \sin^{-1} \sqrt{\frac{x}{a + x}}\, dx$

(vi) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$

(vii) $\displaystyle\int \frac{\sin x}{\sin 4x}\, dx$

(viii) $\displaystyle\int_{-1}^{3/2} |x \sin \pi x|\, dx$

(ix) $\displaystyle\int \frac{\sin(x-a)}{\sin(x+a)}\, dx$

(x) $\displaystyle\int \frac{x^2}{(x^2+4)(x^2+9)}\, dx$

(xi) $\displaystyle\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$

Long Answer Type Questions (6 Marks)

59. Evaluate the following integrals:

(*i*) $\displaystyle\int \frac{x^5+4}{x^5-x}\, dx$

(ii) $\displaystyle\int \frac{2e^t}{e^{3t} - 6e^{2t} + 11e^t - 6}\, dt$

(iii) $\displaystyle\int \frac{2x^3}{(x+1)(x-3)^2}\, dx$

(iv) $\displaystyle\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\, dx$

(v) $\int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

(vi) $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

(vii) $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

60. Evaluate the following integrals as limit of sums:

(i) $\int_2^4 (2x+1)\,dx$

(ii) $\int_0^2 (x^2+3)\,dx$

(iii) $\int_1^3 (3x^2-2x+4)\,dx$

(iv) $\int_0^4 (3x^2+e^{2x})\,dx$

(v) $\int_0^1 e^{2-3x}\,dx$

(vi) $\int_0^1 (3x^2+2x+1)\,dx$

61. Evaluate:

(i) $\int \frac{dx}{(\sin x - 2\cos x)(2\sin x + \cos x)}$

(ii) $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$

(iii) $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\, dx$

62. $\int_0^1 x(\tan^{-1} x)^2\, dx$

63. $\int_0^{\pi/2} \log\sin x\, dx$

64. *Prove that* $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x\, dx$

Hence or otherwise evaluate the integral $\int_0^1 (1-x+x^2)\, dx.$

65. *Evaluate* $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx.$

Answers

1. $\frac{\pi}{2}x + c$

2. $2e - 2$

3. $\tan x + c$

4. $\frac{x^2}{2a} + a|\log x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + \frac{ax^2}{2} + c$

5. 0

6. $\log|\log|\log x|| + c$

7. 0

8. $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + c$

9. $\tan x + c$

10. 0

11. $\frac{(x-2)\sqrt{x^2-4x+10}}{2} + 3\log\left|(x-2) + \sqrt{x^2-4x+10}\right| + c$

12. 0

13. $\tan x - \cot x + c$

14. $3\log_e 2$

15. $\log|x| + c$

16. $\frac{\left(\frac{e}{a}\right)^x}{\log\left(\frac{e}{a}\right)} + c$

17. $\frac{2^x e^x}{\log(2e)} + c$

18. $\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + c$

19. $\log|x+1| + \frac{1}{x+1} + c$

20. $2e^{\sqrt{x}} + c$

21. $x\cos^2\alpha + c$

22. $\frac{\log|x\ \cos\alpha + 1|}{\cos\alpha} + c$

23. $\frac{(\log|\sec x + \tan x|)^2}{2} + c$

24. $\frac{\log|\cos\alpha + x\sin\alpha|}{\sin\alpha} + c$

25. $\tan|\log x| + c$

26. $\log\left|e^x + \sqrt{4 + e^{2x}}\right| + c$

27. $\frac{1}{3}\log|2 + 3\log x| + c$

28. $\log|x + \cos x| + c$

29. $2\log\left|\sec\frac{x}{2}\right| + c$

30. $\frac{1}{e}\log|x^e + e^x| + c$

31. $\frac{(x+\log x)^2}{2} + c$

32. $a\,\frac{x^2}{2} + \frac{\log|x|}{a} - 2x + c$

33. 0

34. 1

35. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$

36. $\frac{b-a}{2}$

37. -1

38. 0

39. $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$

40. 0

41. $x + \log x + c$

42. $\frac{1}{2}\log|\sec x + \tan x| + c$

43. $-\frac{1}{2}\left(\frac{\sin 3x}{3} - \sin x\right) + c$ or $\frac{2}{3}\sin^3 x + c$

44. $2-\sqrt{2}$

45. $\frac{2}{3}\left[(x+2)^{3/2} - (x+1)^{3/2}\right] + c$

46. $\log|1 + \sin x| + c$

47. $x - \sin x + c$

48. $\log|\cos x + \sin x| + c$

49. $\frac{\left(\frac{a}{c}\right)^x}{\log|{}^a/_c|} + \frac{\left(\frac{b}{c}\right)^x}{\log|{}^b/_c|} + c_1$

50. 0

51.

(I) $\frac{1}{2}\log\left[\operatorname{cosec}(\tan^{-1} x^2) - \frac{1}{x^2}\right] + c$

(II) $\frac{1}{2}\left(x^2 - x\sqrt{x^2-1}\right) + \frac{1}{2}\log\left|x + \sqrt{x^2-1}\right| + c$

(III) $\frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c$

(IV) $x\cos 2a - \sin 2a\ \log|\sec(x-a)| + c$

(V) $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

(VI) $\frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| + c$

(VII) $\frac{1}{32}\left[2x + \frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x - \frac{1}{6}\sin 6x\right] + c$

(VIII) $-\left(\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4}\right) + c$

(IX) $\frac{1}{a^2-b^2}\sqrt{a^2\sin^2 x + b^2\cos^2 x} + c$

(X) $-2\text{cosec}\, a\sqrt{\cos a - \tan x\ \sin a} + c$

(XI) $\tan x - \cot x - 3x + c$

(XII) $\sin^{-1}[\sin x - \cos x] + c$

52. (I) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$

(II) $\log\left|\frac{2\log x+1}{3\log x+2}\right| + c$

(III) $\frac{-2}{\sqrt{\tan x}} + \frac{2}{3}\tan^{3/2}x + c$

(IV) $\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{1}{\sqrt{2}}\left(x - \frac{1}{x}\right)\right\} + c$

(V) $2\log\left|\sqrt{x-a} + \sqrt{x-b}\right| + c$

(VI) $\frac{5}{6}\log|3x^2+2x+1|+\frac{-11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c$

(VII) $x-3\log|x^2+6x+12|+2\sqrt{3}\ \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)+c$

(VIII) $-\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+c$

(IX) $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

(X) $\frac{\tan^5 x}{5}+\frac{\tan^7 x}{7}+c$

(XI) $-\log\left|\cos x+\frac{1}{2}+\sqrt{\cos^2 x+\cos x}\right|+c$

53. (I) $\frac{1}{7}\log\left|\frac{x^7}{x^7+1}\right|+c$

(II) $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+c$

(III) $\frac{-2}{3}\log|\cos\theta-2|-\frac{1}{3}\log|1+\cos\theta|+c$

(IV) $\frac{1}{14}\log\left|\frac{x^2+3}{(2-x)^2}\right|+\frac{2}{7\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

(V) $x+4\log\left|\frac{(x-2)^2}{x-1}\right|+c$

(VI) $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\ \tan^{-1}\left(\frac{x}{2}\right)+c$

(VII) $\frac{2}{17}\log|2x+1|-\frac{1}{17}\log|x^2+4|+\frac{1}{34}\tan^{-1}\frac{x}{2}+c$

(VIII) $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$

(IX) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + c$

(X) $-\frac{1}{2}\log|\cos x - 1| - \frac{1}{6}\log|\cos x + 1| + \frac{2}{3}\log|1 - 2\cos x| + c$

54.

(I) $\frac{1}{3}[-x^3 \cos x^3 + \sin x^3] + c$

(II) $\frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + c$

(III) $\frac{e^{ax}}{a^2+b^2}[a\cos(bx + c) + b\sin(bx + c)] + c$

(IV) $2x\ \tan^{-1} 3x - \frac{1}{3}\log|1 + 9x^2| + c$

(V) $2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

(VI) $\left(\frac{x^4-1}{4}\right)\tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + c$

(VII) $\frac{1}{2}e^{2x}\tan x + c$

(VIII) $\frac{x}{\log x} + c$

(IX) $\left(\frac{x-a}{2}\right)\sqrt{2ax - x^x} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$

(X) $e^x\left(\frac{x-1}{x+1}\right) + c$

(XI) $\frac{x^4}{4}\sin^{-1}\left(\frac{1}{x}\right)+\frac{x^2+2}{12}\sqrt{x^2-1}+c$

(XII) $x\log|\log x|-\frac{x}{\log x}+c$

(XIII) $-2(6+x+x^2)^{3/2}+8\left[\frac{2x-1}{4}\sqrt{6+x-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{2x-1}{5}\right)\right]+c$

(XIV) $\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

(XV) $x\tan^{-1}x-\frac{1}{2}\log|1+x^2|-x\tan^{-1}5+c$

(XVI) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

55. (I) $\frac{1}{20}\log 3$

(II) $-\pi/4$

(III) $\frac{\pi}{4}-\frac{1}{2}$

(IV) $\frac{\pi}{4}-\frac{1}{2}\log 2$

(V) $\frac{\pi}{2}$

(VI) $\pi/4$

(VII) $\pi/2$

(VIII) $\frac{3}{4}+\frac{3}{2}\log\frac{2}{3}$

(IX) $\frac{3}{2\pi} - \frac{1}{\pi^2}$

(X) $2\pi + \frac{1}{2a}\sin 2a\pi - \frac{1}{2b}\sin 2b\pi$

56. (I) $\frac{1}{2}$

(II) π

(III) $e^{\pi/4} + e^{-\pi/4}$

(IV) $\frac{1}{4}\pi^2$

(IX) $5 - \sqrt{3} - \sqrt{2}$

(X) $\frac{\pi^2}{16}$

(XI) $\frac{\pi^2}{2ab}$

57. (I) $\frac{\pi}{12}$

(II) 2

(III) $\frac{\pi}{2}$

(IV) $\frac{\pi^2}{4}$

(V) $a\pi$

58. (I) $\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2\sqrt{x-x^2}}{\pi}-x+c$

(II) $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+c$

(III) $\frac{x-2}{x+2}e^x+c$

(IV) $\frac{\sin x-x\cos x}{x\sin x+\cos x}+c$

(V) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+c$

(VI) $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

(VII) $\frac{1}{8}\log\left|\frac{1-\sin x}{1+\sin x}\right|-\frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right|+c$

(XII) $\frac{3}{\pi}+\frac{1}{\pi^2}$

(XIII) $(\cos 2a)(x+a)-(\sin 2a)\log|\sin(x+a)|+c$

(XIV) $-\frac{4}{5}\log|x^2+4|+\frac{9}{5}\log|x^2+9|+c$

(XV) $-\left(\frac{1}{2}\sin 2x+\sin x\right)+c$

59. (I) $x-4\log|x|+\frac{5}{4}\log|x-1|+\frac{3}{4}\log|x+1|+\log|x^2+$
$1|\frac{-1}{2}\tan^{-1}x+c$

(II) $\log\left|\frac{(e^t-1)\,(e^t-3)}{(e^t-2)^2}\right|+c$

(III) $2x-\frac{1}{8}\log|x+1|+\frac{81}{8}\log|x-3|-\frac{27}{2(x-3)}+c$

(IV) $\frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right|+\frac{1}{2(1+\cos x)}+\tan\frac{x}{2}+c$

(V) $\pi/2$

(VI) $\frac{\pi-2}{4}$

(I) $\frac{\pi}{4}-\frac{1}{2}\log 2$

60. (I) 14

(II) $\frac{26}{3}$

(III) 26

(IV) $\frac{1}{2}(127+e^8)$

(V) $\frac{1}{3}\left(e^2-\frac{1}{e}\right)$

(VI) 3

61. (I) $\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right|+c$

(II) $\frac{\pi}{8}\log 2$

(III) $\frac{\pi}{2}\log\frac{1}{2}$

62. $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$

63. $\frac{-\pi}{2}\log 2$

64. $\log 2$

65. $\frac{1}{\sqrt{2}}\log\left|\sqrt{2}+1\right|$

CHAPTER 8

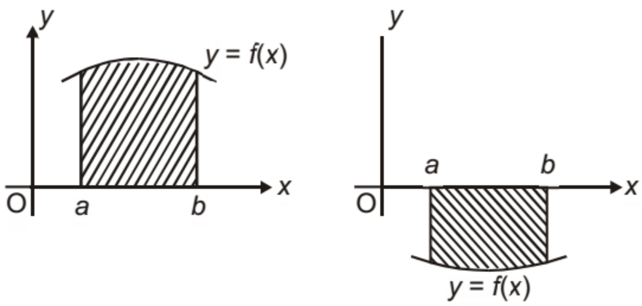
APPLICATIONS OF INTEGRALS

POINT TO REMEMBER

AREA OF BOUNDED REGION

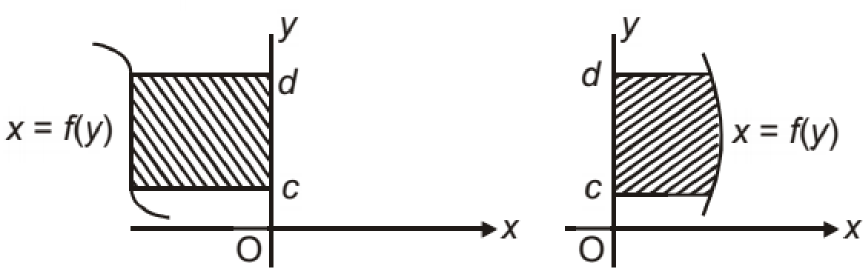
- Area bounded by the curve $y = f(x)$, the x axis and between the ordinates, $x = a$ and $x = b$ is given by

$$Area = \left|\int_a^b f(x)dx\right|$$

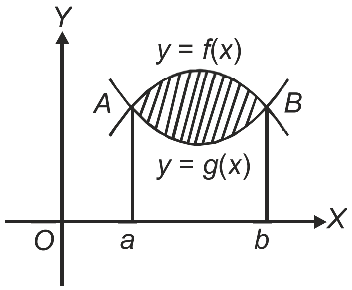


- Area bounded by the curve $x = f(y)$ the y-axis and between abscissas, y = c and y = d is given by

$$Area = \left|\int_c^d f(y)dy\right|$$



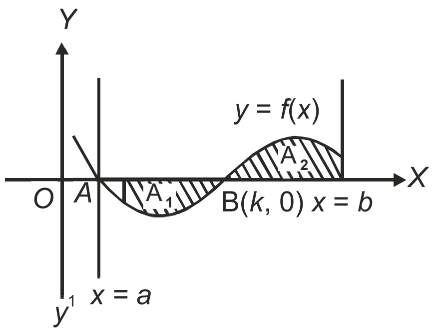
- Area bounded by two curves $y = f(x)$ and $y = g(x)$ such that $0 \le g(x) \le f(x)$ for all $x \in [a, b]$ and between the ordinates $x = a$ and $x = b$ is given by



Area = $\int_a^b [f(x) - g(x)]dx$

- Required Area

$$\left|\int_a^k f(x)dx\right| + \left|\int_k^b f(x)dx\right|.$$



LONG ANSWER TYPE QUESTIONS (6 MARKS)

1. Find the area of the parabola $y^2 = 4ax$ bounded by its Latus rectum.

2. Find the area of the region $\{(x, y): x^2 \le y \le |x|\}$.

3. Find the area of region in the first quadrant enclosed by x–axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

4. Find the area of region $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$

5. Prove that the curve $y = x^2$ and, $x = y^2$ divide the square bounded by $x = 0$, $y = 0$, $x = 1$, $y = 1$ into three equal parts.

6. Find the area of the smaller region enclosed between ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the line $bx + ay = ab$.

7. Find the common area bounded by the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

8. Using integration, find the area of the triangle whose sides are given by $2x + y = 4,\ 3x - 2y = 6\ and\ x - 3y + 5 = 0.$

9. Using integration, find the area of the triangle whose vertices are (–1, 0), (1, 3) and (3, 2).

10. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

11. Find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

12. Find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$.

13. Using integration, find the area enclosed by the curve $y = \cos x$, $y = \sin x$ and x-axis in the interval $[0, \pi/2]$.

14. Using integration, find the area of the following region: $\left\{(x, y) : |x - 1| \leq y \leq \sqrt{5 - x^2}\right\}$

15. Using integration, find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

16. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$ and y-axis.

17. Find the area of the region bounded by the curves $ay^2 = x^3$, the y-axis and the lines y = a and y = 2a.

18. Find the area bounded by x-axis, the curve $y = 2x^2$ and tangent to the curve at the point whose abscissa is 2.

19. Using integration, find the area of the region bounded by the curve $y = 1 + |x + 1|$ and lines $x = -3, x = 3, y = 0$.

20. Find the area of the region $\{(x, y): y^2 \geq 6x,\ \ x^2 + y^2 \leq 16\}$

21. Find the area of the region enclosed between curves $y = |x - 1|$ and $y = 3 - |x|$.

ANSWERS

1. $\frac{8}{3}a^2$ sq. units
2. $\frac{1}{3}$ sq. units
3. 4π sq. units
4. $\left[\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$ sq. units
5. ….
6. $\left(\frac{\pi-2}{4}\right)$ab sq. units
7. $\left(\frac{8\pi}{3} - 2\sqrt{3}\right)$ sq. units
8. 3.5 sq. units
9. 4 sq. units
10. $\left(\pi - \frac{1}{2}\right)$ sq. units
11. $\frac{9}{8}$ sq. units
12. $\frac{4}{3}(8 + 3\pi)$ sq. units
13. $\left(2 - \sqrt{2}\right)$ sq. units
14. $\left(\frac{5\pi}{4} - \frac{1}{2}\right)$ sq. units
15. $2\sqrt{3}$ sq. units
16. $\frac{10}{3}$ sq. units
17. $\frac{3}{5}\ a^2\left[(32)^{\frac{1}{3}} - 1\right]$ sq. units
18. $\frac{4}{3}$ sq. units
19. 16 sq. units
20. $\frac{32\pi - 4\sqrt{3}}{3}$ sq. units
21. 2 sq. units

CHAPTER–9

DIFFERENTIAL EQUATIONS

POINTS TO REMEMBER

- **Differential Equation:** Equation containing derivatives of a dependant variable with respect to an independent variable is called differential equation.

- **Order of a Differential Equation:** The order of a differential equation is defined to be the order of the highest order derivative occurring in the differential equation.

- **Degree of a Differential Equation:** Highest power of highest order derivative involved in the equation is called degree of differential equation where equation is a polynomial equation in differential coefficients.

- **Formation of a Differential Equation:** We differentiate the family of curves as many times as the number of arbitrary constant in the given family of curves. Now eliminate the arbitrary constants from these equations.

After elimination the equation obtained is differential equation.

- **Solution of Differential Equation**

  (i) **Variable Separable Method**

  $\frac{dy}{dx} = f\ (x, y).$

  We Separate the variables and get

  $$f(x)dx = g(y)dy$$

  Then $\int f\ (x)dx = \int g\ (y)dy + c$ is the required solutions.

(ii) **Homogenous Differential Equation:** A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f(x, y) and g(x, y) are both homogenous functions of the same degree in x and y i.e., of the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ is called a homogeneous differential equation.

For solving this type of equations we substitute y=vx and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. The equation can be solved by variables separable method.

(iii) **Linear Differential Equation*:*** An equation of the from $\frac{dy}{dx} + Py = Q$

where *P* and *Q* are constant or functions of *x* only is called a linear differential equation. For finding solution of this type of equations, we find integrating factor (*I.F.*) = $e^{\int P\,dx}$

Solution is *y* (*I.F.*) = $\int Q.(i.F.)dx + c$

Similarly, differential equations of the type $\frac{dx}{dy} + Px = Q$ where P and Q are constants or functions of *y* only can be solved.

Very Short Answer Type Questions (1 Mark)

1. Write the order and degree of the following differential equations.

(i) $\frac{dy}{dx} + \cos y = 0$

(ii) $\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2} = 4$

(iii) $\frac{d^4y}{dx^4} + \sin x = \left(\frac{d^2y}{dx^2}\right)^5$

(iv) $\frac{d^5y}{dx^5} + \log\left(\frac{dy}{dx}\right) = 0$

(v) $\sqrt{1+\frac{dy}{dx}} = \left(\frac{d^2y}{dx^2}\right)^{1/3}$

(vi) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

(vii) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = \sin x$

(viii) $\frac{dy}{dx} + \tan\left(\frac{dy}{dx}\right) = 0$

2. Write the general solution of following differential equations.

(I) $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$

(II) $(e^x + e^{-x})dy = (e^x - e^{-x})dx$

(III) $\frac{dy}{dx} = x^3 + e^x + x^e$

(IV) $\frac{dy}{dx} = 5^{x+y}$

(V) $\frac{dy}{dx} = \frac{1-\cos 2x}{1+\cos 2y}$

(VI) $\frac{dy}{dx} = \frac{1-2y}{3x+1}$

3. Write integrating factor differential equations

(I) $\frac{dy}{dx} + y\cos x = \sin x$

(II) $\frac{dy}{dx} + y\sec^2 x = \sec x + \tan x$

(III) $x^2 \frac{dy}{dx} + y = x^4$

(IV) $x\frac{dy}{dx} + y\log x = x + y$

(V) $x\frac{dy}{dx} - 3y = x^3$

(VI) $\frac{dy}{dx} + y \tan x = \sec x$

(VII) $\frac{dy}{dx} + \frac{1}{1+x^2} y = \sin x$

4. Write order of the differential equation of the family of following curves.

(I) $y = Ae^x + Be^{x+c}$

(II) $Ay = Bx^2$

(III) $(x-a)^2 + (y-b)^2 = 9$

(IV) $Ax + By^2 = Bx^2 - Ay$

(V) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

(VI) $y = a\cos(a+b)$

(VII) $y = a + be^{x+c}$

Short Answer Type Questions (4 Marks)

5. (I) Show that $y = e^{m \sin^{-1} x}$ is a solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

(II) Show that y = sin (sin x) is a solution of differential equation

$$\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} + y\cos^2 x = 0$$

(III) Show that $y = Ax + \frac{B}{x}$ is a solution of $\frac{x^2 d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

(IV) Show that $y = a\ \cos(\log x) + b\ \sin(\log x)$ is a solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

(V) Verify that $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ satisfies the differential equation: $(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

(VI) Find the differential equation of the family of curves

$y = e^x(A\ \cos x + B\sin x)$, where A and B are arbitrary constants.

(VII) Find the differential equation of an ellipse with major and minor axes 2a and 2b respectively.

(VIII) Form the differential equation representing the family of curves $(y - b)^2 = 4(x - a)$.

6. Solve the following differential equations.

(I) $(1-x^2)\frac{dy}{dx} - xy = x^2$, given that $x = 0, y = 2$

(II) $x\frac{dy}{dx} + 2y = x^2 \log x$

(III) $\frac{dy}{dx} + \frac{1}{x}\, y = \cos x + \frac{\sin x}{x}, \quad x > 0$

(IV) $dy = \cos x\,(2 - y \operatorname{cosec} x)dx$; given that $x = \frac{\pi}{2}, y = 2$

(V) $ydx + (x - y^3)dy = 0$

(VI) $ye^y dx = (y^3 + 2xe^y)dy$

7. Solve each of the following differential equations:

(I) $y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right)$

(II) $\cos y\; dx + (1 + 2e^{-x})\sin y\; dy = 0$

(III) $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0$

(IV) $\sqrt{(1-x^2)(1-y^2)}dy + xy\,dx = 0$

(V) $(xy^2 + x)dx + (yx^2 + y)dy = 0; y(0) = 1$

(VI) $\frac{dy}{dx} - y\sin^3 x\cos^3 x + xy\,e^x$

(VII) $\tan x \tan y \, dx + \sec^2 x \sec^2 y \, dy = 0$

(VIII) $\frac{dy}{dx} = x - 1 + xy - y$

8. Solve the following differential equations:

(I) $x^2 y \, dx - (x^3 + y^3) \, dy = 0$

(II) $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

(III) $(x^2 - y^2)dx + 2xy \, dy = 0, \quad y(1) = 1$

(IV) $\left(y \sin\frac{x}{y}\right) dx = \left(x \sin\frac{x}{y} - y\right) dy$

(V) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

(VI) $x \frac{dy}{dx} = y \left(\log y - \log x + 1\right)$

(VII) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

(VIII) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

(IX) $(3xy + y^2)dx + (x^2 + xy)dy = 0$

9. (I) Form the differential equation of the family of circles touching y-axis at (0, 0).

(II) Form the differential equation of family of parabolas having vertex at (0,0) and axis along the (i) positive y-axis (ii) positive x-axis.

(III) Form differential equation of family of circles passing through origin and whose centres lie on x-axis.

(IV) Form the differential equation of the family of circles in the first quadrant and touching the coordinate axes.

10. Show that the differential equation $\frac{dy}{dx} = \frac{x+2y}{x-y}$ is homogeneous and solve it.

11. Show that the differential equation:

$(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$ is homogeneous and solve it.

12. Solve the following differential equations:

(I) $\frac{dy}{dx} - 2y = \cos 3x$

(II) $\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x \; if \; y\left(\frac{\pi}{2}\right) = 1$

(III) $\log\left(\frac{dy}{dx}\right) = px + qy$

13. Solve the following differential equations:

(I) $(x^3 + y^3)\, dx = (x^2y + xy^2)dy$

(II) $x\, dy - y\, dx = \sqrt{x^2 + y^2}dx$

(III) $y\left\{x \cos\left(\frac{y}{x}\right) + y \sin\left(\frac{y}{x}\right)\right\} dx$

$-x\left\{y \sin\left(\frac{y}{x}\right) - x \cos\left(\frac{y}{x}\right)\right\} dy = 0$

(IV) $x^2 dy + y(x + y)\, dx = 0$ given that y=1 when x=1.

(V) $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$ if y (e) = 0

(VI) $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)dy$

(VII) $\frac{dy}{dx} - \frac{y}{x} + \text{cosec}\left(\frac{y}{x}\right) = 0$ given that y = 0 when x = 1

14. Solve the following differential equations:

(I) $\cos^2 x\frac{dy}{dx} = \tan x - y$

(II) $x\ \cos x\ \frac{dy}{dx} + y\ (x\ \sin x + \cos x) = 1$

(III) $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$

(IV) $(y - \sin x)\, dx + \tan x\ dy = 0,\ y(0) = 0$

Long Answer Type Questions (6 Marks)

15. Solve the following differential equations:

(I) $(x\, dy - y\, dx)y\ \sin\left(\frac{y}{x}\right) = (y\, dx + x\, dy)\, x \cos\left(\frac{y}{x}\right)$

(II) $3e^x\ \tan y\ y\, dx + (1 - e^x) \sec^2 y\ dy = 0$ given that $y = \frac{\pi}{4}$, when x = 1

(III) $\frac{dy}{dx} + y \cot x = 2x + x^2\ \cot x$ given that y(0) = 0.

16. Show that the differential equation

$2y\, e^{\frac{x}{y}}\, dx + \left(y - 2x\, e^{\frac{x}{y}}\right) dy = 0$ is homogenous. Find the particular solution of this differential equation given that x = 0 when y = 1.

ANSWERS

1. (i) order = 1, degree is not defined

   (ii) order = 2,degree = 1

   (iii) order = 4,degree = 1

   (iv) order = 5,degree is not defined.

   (v) order = 2, degree = 2

   (vi) order = 2,degree = 2

   (viii) order = 3,degree = 2

   (viii) order = 1,degree is not defined

2. (I) $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + c$

   (II) $y = \log_{e|e^x + e^{-x}|+c}$

   (III) $y = \frac{x^4}{4} + e^x + \frac{x^{e+1}}{e+1} + c$

   (IV) $5^x + 5^{-y} = c$

   (V) $2(y - x) + \sin 2y + \sin 2x = c$

   (VI) $2\log|3x + 1| + 3\log|1 - 2y| = c$

3. (I) $e^{\sin x}$ (II) $e^{\tan x}$

(III) $e^{-1/x}$ (IV) $e^{\frac{(\log x)^2}{2}}$

(V) $\frac{1}{x^3}$ (VI) $\sec x$

(VII) $e^{\tan^{-1} x}$

4. (I) 2 (II) 1

(III) 2 (IV) 1

(V) 1 (VI) 2

(VII) 2

5. (VI) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(VII) $x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx}$

(VI) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

6. (I) $y\sqrt{1-x^2} + \frac{x\sqrt{1-x^2}}{2} = \frac{\sin^{-1} x}{2} + 2$

(II) $y = \frac{x^2(4\log_e x - 1)}{16} + \frac{c}{x^2}$

(III) $y = \sin x + \frac{c}{x}, x > 0$

(IV) $2y\ \sin x = 3 - \cos 2x$

(V) $xy = \frac{y^4}{4} + c$

(VI) $x = -y^2 e^{-y} + cy^2$

7. (I) $cy = (x+2)(1-2y)$

(II) $(e^x + 2)\sec y = c$

(III) $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

(IV) $\frac{1}{2}\log\left|\frac{\sqrt{1-y^2}-1}{\sqrt{1-y^2}+1}\right| = \sqrt{1-x^2} - \sqrt{1-y^2} + c$

(V) $(x^2+1)(y^2+1) = 2$

(VI) $\log y = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + xe^x - e^x + c$

$$= \frac{1}{16}\left[\frac{\cos^3 2x}{3} - \cos 2x\right] + (x-1)e^x + c$$

(VII) $\log|\tan y| - \frac{\cos 2x}{4} = c$

(VIII) $\log|y+1| = \frac{x^2}{2} - x + c$

8. (I) $\frac{-x^3}{3y^3} + 3\log|y| = c$

(II) $\tan^{-1}\left(\frac{y}{x}\right) = \log|x| + c$

(III) $x^2 + y^2 = 2x$

(IV) $y = ce^{\cos(x/y)}$

(V) $\sin\left(\frac{y}{x}\right) = cx$

(VI) $\log|y/x| = cx$

(VII) $-e^{-y} = e^{x} + \frac{x^3}{3} + c$

(VIII) $\sin^{-1} y = \sin^{-1} x + c$

(IX) $|y^2 + 2xy| = \frac{c}{x^2}$

9. (I) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

(II) $2y = x\frac{dy}{dx},\ y = 2x\frac{dy}{dx}$

(III) $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

(IV) $(x - y)^2(1 + y^1)^2 = (x + yy^1)^2$

10. $\log|x^2 + xy + y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3x}}\right) + c$

11. $\frac{x^3}{x^2+y^2} = \frac{c}{x}(x + y)$

12. (I) $y = \frac{3\sin 3x}{13} - \frac{2\cos 3x}{13} + Ce^{2x}$

(II) $y = \frac{2}{3}\sin^2 x + \frac{1}{3}\operatorname{cosec} x$

(III) $-\frac{1}{q}e^{-qy} = \frac{1}{p}e^{px} + c$

13. (I) $-y = x\log\{c(x - y)\}$

(II) $cx^2 = y + \sqrt{x^2 + y^2}$

(III) $xy \cos\left(\frac{y}{x}\right) = c$

(IV) $3x^2 y = y + 2x$

(V) $y = -x \log\ (\log|x|),\ x \neq 0$

(VI) $c(x^2 + y^2) = \sqrt{x^2 - y^2}$

(VII) $\cos\frac{y}{x} = \log|x| + 1$

14. (I) $y = \tan x - 1 + ce^{-\tan x}$

(II) $y = \frac{\sin x}{x} + c\frac{\cos x}{x}$

(III) $x + ye^{\frac{x}{y}} = c$

(IV) $2y = \ \sin x$

15. (I) $c\ xy = \ \sec\left(\frac{y}{x}\right)$

(II) $(1 - e)^3\ \tan y = (1 - e^x)^3$

(III) $y = x^2$

16. $e^{x/y} = -\frac{1}{2}\log|y| + 1$

CHAPTER–10

VECTORS

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a vector. It is denoted by a directed line segment.

- Two or more vectors which are parallel to same line are called collinear vectors.

- Position vector of a point P(a, b, c) w.r.t. origin (0, 0, 0) is denoted by $\overline{OP}$ where $\overline{\mathrm{OP}} = a\hat{i} + b\hat{j} + c\hat{k}$ and $|\overline{OP}| = \sqrt{a^2 + b^2 + c^2}$.

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then

  $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and

  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z)^2}$

- Any vector $\vec{a}$ is called unit vector if $|\vec{a}| = 1$ It is denoted by $\hat{a}$

- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.

- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda\,\vec{a}$ is vector collinear with $\vec{a}$ and $|\lambda\vec{a}| = |\lambda||\vec{a}|$.

- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\vec{b}$ where λ is some scalar.

- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$ where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B, and C is any point which divides $\overline{AB}$ in ratio m:n internally then position vector $\vec{c}$ of point C is given as $\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$. if C divides $\overline{AB}$ in ratio m:n externally, then $\vec{c} = \frac{m\vec{b}-n\vec{a}}{m-n}$.

- The angles α, β and γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, y and z-axis are called angles and cosines of these angles are called direction cosines of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

  Also $l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$

- The numbers a, b, c proportional to l, m, n are called direction ratios.

- Scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}\,\overrightarrow{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$ is the angle between $\vec{a}$ and $\vec{b}$.$(0 \le \theta \le \pi)$.

- Dot product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

- $\overrightarrow{a}.\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{o}, \vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.

- $\vec{a}.\vec{a} = |\vec{a}|^2$, so $\hat{i}.\hat{I} = \hat{j}.\hat{J} = \hat{k}.\hat{K} = 1$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

  $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

- Projection of $\vec{a}$ on $\vec{b} = \left|\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right|$ and

  Projection vector of $\vec{a}$ along $\vec{b} = \left(\frac{(\vec{a}.\vec{b})}{|\vec{b}|}\right)\hat{b}$.

- Cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\,\hat{n}$. were θ is the angle between $\vec{a}$ and $\vec{b}$.$(0 \le \theta \le \pi)$. And $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a} . \vec{b}$ and $\hat{n}$ from a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$ or $\vec{a} \| \vec{b}$.

- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$.

- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$ and $\hat{\jmath} \times \hat{\imath} = \widehat{-k}, \hat{k} \times \hat{\jmath} = \widehat{-\imath}, \hat{\imath} \times \hat{k} = \widehat{-\jmath}$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\bar{a}$ and $\bar{b} = \pm\left(\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}\right)$.

- $|\vec{a} \times \vec{b}|$ is the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

- $\frac{1}{2}|\vec{a} \times \vec{b}|$ is the area of parallelogram where diagonals are $\bar{a}$ and $\bar{b}$.

- If $\bar{a}, \bar{b}$ and $\bar{c}$ form a triangle, then area of the triangle

- $= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|\vec{b} \times \vec{c}\right| = \frac{1}{2}\left|\vec{c} \times \vec{a}\right|.$

- Scalar triple product of three vector $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\bar{a}.\left(\vec{b} \times \vec{c}\right)$ and is denotes as $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

- Geometrically, absolute value of scalar triple product $\left[\vec{a}\vec{b}\vec{c}\right]$ represents volume of a parallelepiped whose coterminous edges are $\vec{a}\,\vec{b}$ and $\vec{c}$.

- $\vec{a}, \bar{b}$ and $\bar{c}$ are coplanar $\Leftrightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$

- $\left[\vec{a}\,\vec{b}\vec{c}\right] = \left[\vec{b}\,\vec{c}\,\vec{a}\,\right] = \left[\vec{c}\,\vec{a}\vec{b}\right]$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then

  $$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Then scalar triple product of three vectors is zero if any two of them are same or collinear.

Very Short Answer Type Questions

1. If $\overrightarrow{AB} = 3\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$ and the coordinate of A are (4,1,1), then find the coordinates of B.

2. Let $\vec{a} = -2\hat{\imath} + \hat{\jmath}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{c} = 4\hat{\imath} + 3\hat{\jmath}$. Find the values of x and y such that $\vec{c} = x\vec{a} + y\vec{b}$.

3. Find a unit vector in the direction of the resultant of the vectors $\hat{\imath} - \hat{\jmath} + 3\hat{k}$, $2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

4. Find a vector of magnitude of 5 units parallel to the resultant of vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$.

5. A vector $\vec{r}$ is inclined to x-axis at 45° and y-axis at 60°. If $|\vec{r}| = 8$ units, find $\vec{r}$.

6. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46$, find $|\vec{a}|$.

7. For what value of λ are the vector $\vec{a}\ and\ \vec{b}$ perpendicular to each other? Where $\vec{a} = \lambda\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{b} = 5\hat{\imath} - 9\hat{\jmath} + 2\hat{k}$

8. Write the projection of $\vec{b} + \vec{c}\ on\ \vec{a}$, where $\vec{a} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ and $\vec{c} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$

9. Write the value of p for which $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are parallel vectors.

10. For any two vectors $\vec{a}\ and\ \vec{b}$ write when $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ holds.

11. Find the value of p if $(2\hat{\imath} + 6\hat{\jmath} + 27\hat{k}) \times (\hat{\imath} + 3\hat{\jmath} + p\hat{k}) = \vec{0}$

12. Evaluate: $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + (\hat{\imath} \times \hat{k}).\hat{\jmath}$

13. If $\vec{a} = 2\hat{\imath} - 3\hat{\jmath}$, $\vec{b} = \hat{\imath} + \hat{\jmath} - \hat{k}$, $\vec{c} = 3\hat{\imath} - \hat{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}\,]$

14. If $\vec{a} = 5\hat{\imath} - 4\hat{\jmath} + \hat{k}$, $\vec{b} = -4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{c} = \hat{\imath} - 2\hat{\jmath} - 2\hat{k}$, then evaluate $\vec{c}.(\vec{a} \times \vec{b})$

15. Show that vector $\hat{\imath} + 3\hat{\jmath} + \hat{k}$, $2\hat{\imath} - \hat{\jmath} - \hat{k}$, $7\hat{\jmath} + 3\hat{k}$ are parallel to same plane.

16. If the points (-1, -1, 2),(2, m, 5) and (3,11,6) are collinear , find the value of m.

17. Find a vector of magnitude 6 which is perpendicular to both the vectors $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $4\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

18. If $\vec{a} \,.\, \vec{b} = 0$, then what can you say about $\vec{a}$ and $\vec{b}$?

19. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, write the value of the following: $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$, then what is the angle between $\vec{a}$ and $\vec{b}$?

21. Find the area of a parallelogram having diagonals $3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

22. If $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are three mutually perpendicular vectors, then find the value of $\hat{\jmath}.(\hat{k} \times \hat{\imath})$.

23. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the segment PQ in the ratio 2:1 externally.

24. Find λ when scalar projection of $\vec{a} = \lambda\hat{\imath} + \hat{\jmath} + 4\hat{k}$ on $\vec{b} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ is 4 units.

25. Find "a" so that the vectors $\vec{p} = 3\hat{\imath} - 2\hat{\jmath}$ and $\vec{q} = 2\hat{\imath} + a\hat{\jmath}$ be orthogonal.

26. If $(\vec{a} \times \vec{b})^2 + (\vec{a}.\vec{b})^2 = 144$ and $|\vec{a}| = 4$ then find the value of $|\vec{b}|$.

27. If $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{c} = \lambda\hat{\imath} - \hat{\jmath} + \lambda\hat{k}$ are coplanar, find the value of λ.

28. What is the point of trisection of PQ nearer to P if positions of P and Q are $3\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ and $9\hat{\imath} + 8\hat{\jmath} - 10\hat{k}$ respectively?

29. What is the angle between $\vec{a}$ and $\vec{b}$, if $\vec{a}\,\vec{b} = 3$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$.

30. The two vectors $\hat{\imath} + \hat{\jmath}$ and $3\hat{\imath} - \hat{\jmath} + 4\hat{k}$ represents the two sides AB and AC respectively of ΔABC. find the length of median through 'A'.

31. If for any two vectors $\vec{a}$ *and* $\vec{b}$, $(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = \lambda\left[(\vec{a})^2 + (\vec{b})^2\right]$, then write the value of λ.

Short Answer Type Questions (4 Marks)

1. The points A,B and C with position vectors $3\hat{\imath} - y\hat{\jmath} + 2\hat{k}$, $5\hat{\imath} - \hat{\jmath} + \hat{k}$ *and* $3x\hat{\imath} + 3\hat{\jmath} - \hat{k}$ are collinear. Find the values of x and y and also the ratio in which the point B divides AC.

2. If sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

3. Let $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$, $\vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ *and* $\vec{b}$ and satisfying $\vec{d}.\vec{c} = 21$

4. If $\hat{a}$ *and* $\hat{b}$ are unit vectors inclined at an angle θ then proved that

   (i) $\cos\frac{\theta}{2} = \frac{1}{2}\,|\hat{a} + \hat{b}|$

   (ii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} - \hat{b}}\right|$

5. If $\vec{a}, \vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that $2\vec{a}+\vec{b}$ is perpendicular to $\vec{b}$.

6. If $\vec{a}, \vec{b}, \vec{c}$ are the three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Also find the angle.

7. Show that the vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} - 3\hat{\jmath} + 5\hat{k}$, $\vec{c} = 2\hat{\imath} + \hat{\jmath} - 4\hat{k}$ form a right angled triangle.

8. For any vector $\vec{a}$ prove that $|\vec{a}\times\hat{\imath}|^2 + |\vec{a}\times\hat{\jmath}|^2 + |\vec{a}\times\hat{k}|^2 = 2|\vec{a}|^2$

9. Show that $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

10. If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of vertices A,B,C of a Δ ABC, show that the area of triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$. Deduce the condition for points $\vec{a}, \vec{b}$ and $\vec{c}$ to be collinear.

11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$ and the angle between b and c is $\pi/6$, prove that $\vec{a} = \pm 2(\vec{b}\times\vec{c})$.

12. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then prove that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

13. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{\jmath} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a}\times\vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$.

14. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. if the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, prove that $[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$

15. If the vectors $\vec{\alpha} = a\hat{\imath} + \hat{\jmath} + \hat{k},\ \vec{\beta} = \hat{\imath} + b\hat{\jmath} + \hat{k}$ and $\vec{\gamma} = \hat{\imath} + \hat{\jmath} + c\hat{k}$ are coplanar, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$ where a$\neq 1, b \neq 1\ and\ c \neq 1$

16. Find the altitude of a parallelepiped determined by the vectors $\vec{a}, \vec{b}\ and\ \vec{c}$ if the base is taken as parallelogram determined by $\vec{a}\ and\ \vec{b}$ and if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k},\ \vec{b} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

17. Show that four points whose position vectors are $6\hat{\imath} - 7\hat{\jmath},\ 16\hat{\imath} - 19\hat{\jmath} - 4\hat{k},\ 3\hat{\imath} - 6\hat{k},\ 2\hat{\imath} - 5\hat{\jmath} + 10\hat{k}$ are coplanar.

18. If $|\vec{a}| = 3,\ |\vec{b}| = 4\ and\ |\vec{c}| = 5$ such that each is perpendicular to sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$

19. Decompose the vector $6\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$.

20. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c},\ \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, a$\neq 0$, then show that $\vec{b} = \vec{c}$.

21. If $\vec{a}, \vec{b}\ and\ \vec{c}$ are three non zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$. Prove that $\vec{a}, \vec{b}\ and\ \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$

22. Simplify $[\vec{a} - \vec{b},\ \vec{b} - \vec{c},\ \vec{c} - \vec{a}]$

23. If $[\vec{a}\ \vec{b}\ \vec{c}] = 2$, find the volume of the parallelepiped whose co-terminus edges are $2\vec{a} + \vec{b},\ 2\vec{b} + \vec{c},\ 2\vec{c} + \vec{a}$.

24. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=5$, $|\vec{b}|=12$, $|\vec{c}|=13$, then find $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

25. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$, find the angle between $\vec{a}$ *and* $\vec{b}$.

26. The magnitude of the vector product of the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ with a unit vector along the sum of the vector $2\hat{\imath}+\widehat{4\jmath}+5\hat{k}$ and $\lambda\hat{\imath}+2\hat{\jmath}+3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

27. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, prove that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

28. Find a vector of magnitude $\sqrt{51}$ which makes equal angles with the vector $\vec{a}=\frac{1}{3}\left(\hat{\imath}-\widehat{2\jmath}+2\hat{k}\right)$, $\vec{b}=\frac{1}{5}\left(-4\hat{\imath}-3\hat{k}\right)$ and $\vec{c}=\hat{\jmath}$

29. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are perpendicular to each other, then prove that $[\vec{a}\ \vec{b}\ \vec{c}]=a^2b^2c^2$

30. If $\overline{\alpha}=3\hat{\imath}-\hat{\jmath}$ and $\overline{\beta}=2\hat{\imath}+\hat{\jmath}+3\hat{k}$ then express $\overline{\beta}$ in the form of $\overline{\beta}=\overline{\beta}_1+\overline{\beta}_2$, where $\overline{\beta}_1$ is parallel to $\overline{\alpha}$ and $\overline{\beta}_2$ is perpendicular to $\overline{\alpha}$.

31. Find a unit vector perpendicular to plane ABC, when position vectors of A,B,C are $\widehat{3\imath}-\hat{\jmath}+2\hat{k}$, $\hat{\imath}-\hat{\jmath}-3\hat{k}$ and $4\hat{\imath}-\widehat{3\jmath}+\hat{k}$ respectively.

32. Find a unit vector in XY plane which makes an angle 45° with the vector $\hat{\imath}+\hat{\jmath}$ at angle of 60° with the vector $3\hat{\imath}-4\hat{\jmath}$.

33. Suppose $\vec{a} = \lambda\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$, $\vec{b} = \lambda\hat{\imath} + \hat{\jmath} + 2\lambda\hat{k}$. If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90°, then prove that λ satisfies the inequality $-7 < \lambda < 1$.

34. Let $\vec{v} = \widehat{2\imath} + \hat{\jmath} - \hat{k}$ and $\vec{w} = \hat{\imath} + 3\hat{k}$. If u is a unit vector, then find the maximum value of he scalar triple products $\vec{u}, \vec{v}, \vec{w}$.

35. If $\vec{a} = \hat{\imath} - \hat{k}$, $\vec{b} = x\hat{\imath} + \hat{\jmath} + (1-x)\hat{k}$ and $\vec{c} = y\hat{\imath} + x\hat{\jmath} + (1+x-y)\hat{k}$ then prove that $[\vec{a}\ \vec{b}\ \vec{c}]$ depends upon neither x nor y.

36. A, b and c are distinct non negative numbers, if the vectors a$\hat{\imath} + a\hat{\jmath} + c\hat{k}$, $\hat{\imath} + \hat{k}$ and $c\hat{\imath} + c\hat{\jmath} + b\hat{k}$ lie in a plane, then prove that c is the geometric mean of a and b.

37. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2), (1, b, b^2)$ *and* $(1, c, c^2)$ are non-coplanar, then find the value of abc. (Ans= –1)

38. If $\vec{x} + \vec{y} + \vec{z} = \vec{0}$, $|\vec{x}| = |\vec{y}| = |\vec{z}| = 2$ and θ is the angle between $\vec{y}$ *and* $\vec{z}$, then find the value of $cosec^2\theta + cot^2\theta$, where $0 \le \theta \le \pi$.

Answers

Very Short Answer

1. (7, 3, 0)

2. $x = -1,\ y = 2$

3. $\frac{1}{\sqrt{21}}(4\hat{\imath} + 2\hat{\jmath} - \hat{k})$

4. $\frac{\sqrt{5}}{2}\,(3\hat{\imath} + \hat{\jmath})$

5. $4\left(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}\right)$

6. 22

7. λ =$\frac{16}{5}$

8. 2

9. $\frac{2}{3}$

10. $\vec{a}\ and\ \vec{b}$ are perpendicular

11. $\frac{27}{2}$

12. 0

13. 4

14. -5

15. ……

16. $m = 8$

17. $-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$

18. Either $\vec{a} = 0\ or\ \vec{b} = 0\ or\ \vec{a} \perp \vec{b}$

19. 0

20. 45°

21. $5\sqrt{3}$ squ. Units

22. 1

23. $-\vec{a} + 4\vec{b}$

24. $\lambda = 5$

25. a = 3

26. 3

27. $\lambda = 1$

28. $\left(5, \frac{14}{3}, -6\right)$

29. $\frac{\pi}{3}$

30. $\frac{\sqrt{34}}{2}$

31. λ = 2

Short Answer Type Answer

1. $x = 3,\ y = 3,\ 1:2$

3. $\vec{d} = 7\hat{\imath} - 7\hat{\jmath} - 7\hat{k}$

6. $\cos^{-1}\frac{1}{\sqrt{3}}$

13. $\vec{b} = \frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$

16. $\frac{4}{\sqrt{38}}$ units

18. $5\sqrt{2}$

19. $\left(-\hat{\imath} - \hat{\jmath} - \hat{k}\right) + \left(7\hat{\imath} - 2\hat{\jmath} - 5\hat{k}\right)$

23. 18 cu. Units

24. -169

25. 60°

26. $\lambda = 1$

30. $\vec{\beta} = \left(\frac{3}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\right) + \left(\frac{1}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} + 3\hat{k}\right)$

31. $\frac{-1}{\sqrt{165}}\left(10\hat{\imath} + 7\hat{\jmath} - 4\hat{k}\right)$

32. $\frac{13}{14}\hat{\imath} + \frac{1}{14}\hat{\jmath}$

37. -1

38. 1

CHAPTER–11

THREE-DIMENSIONAL GEOMETRY

POINTS TO REMEMBER

- **Distance Formula:** Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** line segment AB is divided by P (x, y, z) in ratio m:n

(a) Internally	(b) Externally
$\left(\frac{m\,x_2 + n\,x_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

- **Direction ratio** of a line through $(x_1, y_1, z_1)$and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

- **Direction cosines** of a line having direction ratios as a, b, c are:

  $$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of line in space:**

Vector form	Cartesian form
(i) Passing through point $\vec{a}$ and parallel to vector $\vec{b}$; $\vec{r} = \vec{a} + \lambda\vec{b}$	(i) Passing through point (x_1, y_1, z_1) and having direction ratios a,b,c; $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

(ii) Passing through two points $\vec{a}$ and $\vec{b}$; $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$	(ii) Passing through two points $(x_1, y_1\ z_1)$ and $(x_2, y_2\ z_2)$; $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Angle between two lines:**

Vector form	Cartesian form
(i) For lines $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ $\cos\theta = \frac{\lvert\vec{b}_1 . \vec{b}_2\rvert}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}$	(ii) For lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ $\cos\theta = \frac{\lvert a_1a_2 + b_1b_2 + c_1c_2\rvert}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
(iii) Lines are perpendicular if $\vec{b}_1 . \vec{b}_2 =$	(ii) Lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
(iv) Lines are parallel if $\vec{b}_1 = k\vec{b}_2$; $k \neq 0$	(iii) Lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Equation of plane:**

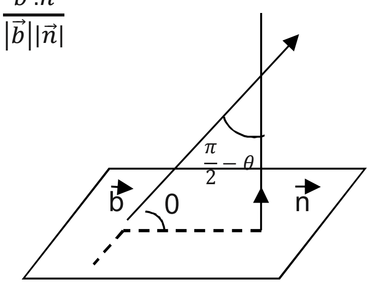
If p is length of perpendicular from origin to plane and $\hat{n}$ is unit vector normal to plane $\vec{r} . \hat{n} = p$	If p is length of perpendicular from origin to plane and l, m, n are d.c.s of normal to plane $lx + my + nz = p$
Passing through $\vec{a}$ and $\vec{n}$ is normal to plane : $(\vec{r} - \vec{a}).\vec{n} = 0$	Passing through (x_1, y_1, z_1) and a, b, c are d.r.s of normal to plane: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Passing through three non collinear points $\vec{a}, \vec{b}, \vec{c}$: $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})\right] = 0$	Passing through three non collinear points $(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)$: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$
If a, b, c are intercepts on co-ordinate axes $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$	If x_1, y_1, z_1 are intercepts on coordinate axes $\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1$
Plane passing through line of intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ (λ=real no.)	Plane passing through the line of intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

- **Angle between planes:**

Angle θ between planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\cos\theta = \frac{\lvert\vec{n}_1 . \vec{n}_2\rvert}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert}$	Angle θ between planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $\cos\theta = \frac{\lvert a_1a_2 + b_1b_2 + c_1c_2\rvert}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
Planes are perpendicular iff $\vec{n}_1 . \vec{n}_2 = 0$	Planes are perpendicular iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$
Planes are parallel iff $\vec{n}_1 = \lambda\,\vec{n}_2$; λ≠ 0	Planes are parallel iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Angle between line and plane:**

Angle θ between line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}\,.\vec{n} = d$ is $\sin\theta = \cos(90^\circ - \theta)$ $= \frac{\vec{b}\,.\vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert}$ 	Angle θ between line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and plane $a_2x + b_2y + c_2z = d$ is $\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$

- **Distance of a point from a plane**

The perpendicular distance p from the point P with position vector $\vec{a}$ to the plane $\vec{r}\,.\vec{n} = d$ is given by $p = \frac{\lvert\vec{a}\,.\vec{n} - d\rvert}{\lvert\vec{n}\rvert}$	The perpendicular distance p from the point P (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is given by $p = \frac{\lvert Ax_1 + By_1 + Cz_1 + D\rvert}{\sqrt{A^2+B^2+C^2}}$

- **Coplanarity**

Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar iff $(\vec{a}_2 - \vec{a}_1)\,.\left(\vec{b}_1 \times \vec{b}_2\right) = 0$	Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar iff $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Shortest distance between two skew lines**

The shortest distance between lines	The shortest distance between
$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $$d = \left\|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{\|\vec{b}_1\times\vec{b}_2\|}\right\|$$	$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$$ Where $D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$

Very Short Answer Type Questions (1 Mark)

1. What is the distance of point (a, b, c) from x-axis?

2. What is the angle between the lines $2x = 3y = -z$ and $6x =- y = -4z$?

3. Write the equation of a line passing through (2, -3, 5) and parallel to line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$.

4. Write the equation of a line through (1, 2, 3) and perpendicular to $\bar{r}.\left(\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) = 5$.

5. What is the value of λ for which the lines $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$ are perpendicular to each other?

6. If a line makes angle $\alpha, \beta \text{ and } \gamma$ with co-ordinate axes, then what is the value of $sin^2\,\alpha + sin^2\,\beta + sin^2\,\gamma$?

7. Write line $\overline{r} = (\hat{\imath} - \hat{\jmath}) + \lambda\,(2\hat{\jmath} - \hat{k})$ into Cartesian form.

8. If the direction ratios of a line are 1, -2, 2 then what are the direction cosines of the line?

9. Find the angle between the planes $2x - 3y + 6z = 9$ and xy – plane.

10. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.

11. What is the perpendicular distance of plane $2x - y + 3z = 10$ from origin?

12. What is the y-intercept of the plane $x - 5y + 7z = 10$?

13. What is the distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$.

14. What is the equation of the plane which cuts off equal intercepts of unit length on the coordinate axes?

15. Are the planes $x + y - 2z + 4 = 0$ and $3x + 3y - 6z + 5 = 0$ intersecting?

16. What is the equation of the plane through the point (1, 4, -2) and parallel to the plane$-2x + y - 3z = 7$?

17. Write the vector equation of the plane which is at a distance of 8 units from the origin and is normal to the vector $(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

18. What is equation of the plane if the foot of perpendicular from origin to this plane is (2, 3, 4)?

19. Find the angles between the planes $\vec{r}.(\hat{\imath} - 2\hat{\jmath} - 2\hat{k}) = 1$ and $\vec{r}.(3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}) = 0$.

20. What is the angle between the line $\frac{x+1}{3} = \frac{2y-1}{4} = \frac{2-z}{-4}$ and the plane $2x + y - 2z + 4 = 0$?

21. If O is origin OP = 3 with direction ratios proportional to -1, 2, -2 then what are the coordinates of P?

22. What is the distance between the line $\overline{r} = 2\hat{\imath} - 2\hat{\jmath} + 3\hat{k} + \lambda\,(\hat{\imath} + \hat{\jmath} + 4\hat{k})$ from the plane $\overline{r}\,.(-\hat{\imath} + 5\hat{\jmath} - \hat{k}) + 5 = 0$.

23. Write the line $2x = 3y = 4z$ in vector form.

24. The line $\frac{x-4}{1} = \frac{2y-4}{2} = \frac{k-z}{-2}$ lies exactly in the plane $2x - 4y + z = 7$. Find the value of k.

Short Answer Type Questions (4 Marks)

25. Find the equation of a plane containing the points (0, –1, –1), (–4, 4, 4) and (4, 5, 1). Also show that (3, 9, 4) lies on that plane.

26. Find the equation of the plane which is perpendicular to the plane $\overline{r}\,.(5\hat{\imath} + 3\hat{\jmath} + 6\hat{k}) + 8 = 0$ and which is containing the line of intersection of the planes $\overline{r}\,.(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = 4$ and $\overline{r}\,.(2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5 = 0$.

27. Find the distance of the point (3, 4, 5) from the plane $x + y + z = 2$ measured parallel to the line $2x = y = z$.

28. Find the distance of the point (-2, 3, -4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

29. Find vector and Cartesian equation of a line passing through a point with position vector $2\hat{\imath} - \hat{\jmath} + \hat{k}$ and which is parallel to the line joining the points with position vectors $-\hat{\imath} + 4\hat{\jmath} + \hat{k}$ and $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

30. Find the equation of the plane passing through the point (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x - 5y = 15$.

31. Find equation of plane through line of intersection of planes $\overline{r}\,.(2\hat{\imath} + 6\hat{\jmath}) + 12 = 0$ and $\overline{r}\,.\left(3\hat{\imath} - \hat{\jmath} + 4\hat{k}\right) = 0$ which is at a unit distance from origin.

32. Find the image of point (3, –2, 1) in the plane $3x - y + 4z = 2$.

33. Find the equation of a line passing through (2, 0, 5) and which is parallel to line $6x - 2 = 3y + 1 = 2z - 2$.

34. Find image (reflection) of the point (7, 4, –3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

35. Find equation of a plane passing through the points (2, –1, 0) and (3,–4, 5) and parallel to the line $2x = 3y = 4z$.

36. Find the distance of the point (–1, –5, –10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ and the plane $x - y + z = 5$.

37. Find the equation of the plane passing through the points (2, 3, -4) and (1, -1, 3) and parallel to the x-axis.

38. Find the distance of the point (1, -2, 3) from the plane $x - y + z = 5$, measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

39. find the equation of he plane passing through the intersection of two plane $3x - 4y + 5z = 10, 2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

40. Find the distance between the planes $2x + 3y - 4z + 5 = 0$ and $\bar{r}\,.\left(4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}\right) = 11$.

41. Find the equation of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ whose perpendicular distance from the point (1, 2, 3) is 1 unit.

42. Show that the lines line $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and line $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect each other. Find the point of intersection.

43. Find the shortest distance between the lines:

$$\bar{r} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} + \mu\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) \text{ and}$$

$$\bar{r} = \left(2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}\right) + \lambda\left(3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}\right).$$

44. Find the distance of the point (-2, 3, -4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

45. Find the equation of plane passing through the point (-1, -1, 2) and perpendicular to each of the plane

$\bar{r}.(2\hat{\imath}+3\hat{\jmath}-3\hat{k})=2$ and $\bar{r}.(5\hat{\imath}-4\hat{\jmath}+\hat{k})=6$

46. Find the equation of a plane passing through (-1, 3, 2) and parallel to each of the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$

47. Show that the plane $\bar{r}.(\hat{\imath}-3\hat{\jmath}+5\hat{k})=7$ contains the line

$\bar{r}=(\hat{\imath}+3\hat{\jmath}+3\hat{k})+\lambda(3\hat{\imath}+\hat{\jmath})$.

Long Answer Type Questions (6 Marks)

48. Check the co planarity of lines

$\bar{r}=(-3\hat{\imath}+\hat{\jmath}+5\hat{k})+\lambda(-3\hat{\imath}+\hat{\jmath}+5\hat{k})$.

$\bar{r}=(-\hat{\imath}+2\hat{\jmath}+5\hat{k})+\mu(-\hat{\imath}+2\hat{\jmath}+5\hat{k})$

If they are coplanar, find equation of the plane containing the lines.

49. Find shortest distance between the lines:

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

50. Find the shortest distance between the lines:

$\bar{r}=(1-\lambda)\hat{\imath}+(\lambda-2)\hat{\jmath}+(3-2\lambda)\hat{k}$

$\bar{r}=(\mu+1)\hat{\imath}+(2\mu-1)\hat{\jmath}-(2\mu+1)\hat{k}$

51. A variable plane is at a constant distance 3 p from the origin and meets the coordinates axes in A, B and C. If the centroid of $\Delta\,ABC$ is (α, β, γ), then show that $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = p^{-2}$

52. A vector $\vec{n}$ of magnitude 8 units is inclined to x-axis at 45°, y axis at 60° and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$and is normal to $\vec{n}$, find its equation in vector form.

53. Find the foot of perpendicular from the point $2\hat{\imath} - \hat{\jmath} + 5\hat{k}$ on the line $\bar{r} = (11\hat{\imath} - 2\hat{\jmath} - 8\hat{k}) + \lambda(10\hat{\imath} - 4\hat{\jmath} - 11\hat{k})$. Also find the length of the perpendicular.

54. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonal of a cube. Prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3}$

55. Find the equation of the plane passing through the intersection of planes $2x + 3y - z = -1$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z = 4$. Also find the inclination of this plane with xy-plane.

56. Find the length and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

57. Show that $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{2}, z = 2$. do not intersect each other.

58. Find equation of plane passing through the foot of perpendiculars drawn from (5, -7, 9) to xy-plane, yz-plane and zx-plane.

Answers

1. $\sqrt{b^2 + c^2}$
2. 90°
3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$
4. $\bar{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + 3\hat{k})$
5. λ = 2
6. 2
7. $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$
8. $\pm\frac{1}{3}, \mp\frac{2}{3}, \pm\frac{2}{3}$
9. $\cos^{-1}(6/7)$
10. $\frac{x}{a} = \frac{y-1}{a} = \frac{z-2}{a}$,

    $a \in R - \{0\}$
11. $\frac{10}{\sqrt{14}}$
12. -2
13. $\frac{1}{6}$
14. $x + y + z = 1$
15. No
16. $-2x + y - 3z = 8$
17. $\bar{r}\,.(2\hat{\imath} + \hat{\jmath} + 2\hat{k}) = 24$
18. $2x + 3y + 4z = 29$
19. $\cos^{-1}\left(\frac{11}{21}\right)$
20. 0 (line is parallel to plane)
21. (–1, 2,–2)
22. $\frac{10}{3\sqrt{3}}$
23. $\vec{r} = \vec{0} + \lambda(6\hat{\imath} + 4\hat{\jmath} + 3\hat{k})$.
24. k = 7
25. $5x - 7y + 11z + 4 = 0$
26. $\bar{r}.(-51\hat{\imath} - 15\hat{\jmath} + 50\hat{k}) = 173$
27. 6 units

28. $\frac{17}{2}$ unit

29. $\bar{r} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

30. $5x + 2y - 3z - 17 = 0$

31. $\bar{r}.(2\hat{\imath} + \hat{\jmath} + 2\hat{k}) + 3 = 0$ or $\bar{r}.(-\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) + 3 = 0$

32. (0, -1, -3)

33. $\frac{x-2}{1} = \frac{y}{2} = \frac{z-5}{3}$

34. $\left(-\frac{51}{7}, -\frac{18}{7}, \frac{43}{7}\right)$

35. $29x$ –27y–22z = 85

36. 13

37. $7y + 4z = 5$

38. 1 unit

39. $x - 20y + 27z = 14$

40. $\frac{21}{2\sqrt{29}}$ units

41. $x - 2y + 2z = 0$ *and* $x - 2y + 2z = 6$

42. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

43. $\frac{1}{\sqrt{6}}$

44. $\frac{17}{2}$ units

45. $\bar{r}.(9\hat{\imath} + 17\hat{\jmath} + 23\hat{k}) = 20$

46. $2x - 7y + 4z + 15 = 0$

47. ..

48. $x - 2y + z = 0$

49. 14 units

50. $\frac{8}{\sqrt{29}}$

51. ..

52. $\bar{r}.(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}) = 2$

53. $(1, 2, 3), \sqrt{14}$

54. ..

55. $7x + 13y + 4z = 9$, $\cos^{-1}\left(\frac{4}{\sqrt{234}}\right)$

56. $SD = 14\ units,$

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

57. ..

58. $63x - 45y + 35z = 630$

CHAPTER 12

LINEAR PROGRAMMING

POINTS TO REMEMBER

- Linear programming is the process used to obtain minimum or maximum value of the linear objectives function under known linear constraints.
- **Objective Functions:** Linear function z = ax + by where a and b are constants, which has to be maximized or minimized is called a linear objective function.
- **Constraints:** the linear inequalities or inequations or restrictions on the variables of a linear programming problem.
- **Feasible Region:** It is defined as a set of points which satisfy all the constraints.
- **To Find Feasible Region:** Draw the graph of all the linear in equations and shade common region determined by all the constraints.
- **Feasible Solutions:** Points within and on the boundary of the feasible region represents feasible solutions of the constraints.
- **Optimal Feasible Solution:** Feasible solution which optimizes the objective function is called optimal feasible solution.

Long Answer Type Questions (6 Marks)

1. Solve the following L.P.P. graphically

Minimise and maximise $z = 3x + 9y$

Subject to the constraints $x + 3y \leq 60$

$x + y \geq 10$

$x \leq y$

$x \geq 0, y \geq 0$

2. Determine graphically the minimum value of the objective function $z = -50x + 20y$, subject to he constraints.

$2x - y \geq -5$

$3x + y \geq 3$

$2x - 3y \leq 12$

$x \geq 0,\ y \geq 0$

3. Two tailors A and B earn Rs. 150 and Rs. 200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost? Solve the problem graphically.

4. There are two types of fertilisers A and B. A consists of 10% nitrogen and 6% phosphoric acid and B consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If A costs Rs. 6 per kg and B costs Rs. 5 per kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at minimum cost. What is the minimum cost? What values are being promoted here?

5. A man has Rs. 1500 to purchase two types of shares of two different companies S_1 and S_2. Market price of one share of S_1 is Rs. 180 and S_2 is Rs 120. He wishes to purchase a maximum of ten shares only. If one share of type S_1 gives a yield of Rs. 11 and of type S_2 yields Rs. 8 then how much shares of each type must be purchased to get maximum profit? And what will be the maximum profit?

6. A company manufactures two types of lamps say A and B. Both lamps go through a cutter and then a finisher. Lamp A requires 2 hours of the cutter's time and 1 hours of the finisher's time. Lamp B requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 100 hours and finisher has 80 hours of time available each month. Profit on one lamp A is Rs. 7.00 and on one lamp B is Rs. 13.00. Assuming that he can sell all that he produces, how many of each type of lamps should be manufactured to obtain maximum profit?

7. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for atmost 20 items. A fan and sewing machine cost Rs. 360 and Rs. 240 respectively. He can sell a fan at a profit of Rs. 22 and sewing machine at a profit of Rs. 18. Assuming that he can sell whatever he buys, how should he invest his money to maximise his profit?

8. If a young man rides his motorcycle at 25 km/h, he has to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/h, the petrol cost increase to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to cover the maximum distance within one hour. Express this as L.P.P. and then solve it graphically.

9. A producer has 20 and 10 units of labour and capital respectively which he can use to produce two kinds of goods X and Y. To produce one unit of X, 2 units of capital and 1 unit of labour is required. To produce one unit of Y, 3 units of labour and 1 unit of capital is required. If X and Y are priced at Rs. 80 and Rs. 100 per unit respectively, how should the producer use his resources to maximise the total revenue?

10. A factory owner purchases two types of machines A and B for his factory. The requirements and limitations for the machines are as follows:

Machine	Area Occupied	Labour Force	Daily Output (In units)
A	1000 m^2	12 men	50
B	1200 m^2	8 men	40

He has maximum area of 7600 m^2 available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

11. A manufacturer makes two types of cups A and B. Three machines are required to manufacture the cups and the time in minutes required by each in as given below:

Types of Cup	**Machines**		
	I	**II**	**III**
A	12	18	6
B	6	0	9

Each machine is available for a maximum period of 6 hours per day. If the profit on each cup A is 75 paisa and on B is 50 paisa, find how many cups of each type should be manufactures to maximise the profit per day.

12. A company produces two types of belts A and B. Profits on these belts are Rs. 2 and Rs. 1.50 per belt respectively. A belt of type A requires twice as much time as belt of type B. The company can produce at most 1000 belts of type B per day. Material for 800 belts per day is available. At most 400 buckles for belts of type A and 700 for type B are available per day. How much belts of each type should the company produce so as to maximize the profit?

13. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on

each second class ticket. The airline reserves at a least 20 seats for first class. However at least four times as many passengers prefer to travel by second class than by first class. Determine how many tickets of each type must be sold to maximize profit for the airline.

14. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of Rs. 5 and Rs. 4 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of minerals and 40 units of calories whereas one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the food A and B should be used to have least cost but it must satisfy the requirements of the sick person. What is balanced diet and what is the importance of balanced diet in daily life?

15. Anil wants to invest at most Rs, 12000 in bonds A and B. According to the rules, he has to invest at least Rs. 2000 in Bond A and at least Rs. 4000 in bond B. If the rate of interest on bond A and B are 8% and 10% per annum respectively, how should he invest this money for maximum interest? Formulate the problem as L.P.P. and solve graphically.

Answers

1. Min $z = 60$ at $x = 5$, $y = 5$

   Max $z = 180$ at the two corner points (0, 20) and (15, 5).

2. No minimum value

3. Minimum cost = Rs. 1350 at 5 days of A and 3 days of B.

4. 100 kg of fertiliser A and 80 kg of fertilisers B; minimum cost Rs. 1000. Values promoted are keeping the productivity of the soil so that vegetables and fruits are free from chemicals.

5. Maximum Profit = Rs. 95 with 5 shares of each type.

6. Lamps of type A = 40, Lamps of type B = 20.

7. Fan: 8; Sewing machine: 12, Maximum Profit = Rs. 392.

8. At 25 km/h he should travel 50/3 km, at 40 km/h, 40/3 km. Maximum distance 30 km in 1 hr.

9. X: 2 units; Y: 6 units; Maximum revenue Rs. 760.

10. Type A: 4; Type B: 3

11. Cup A: 15; Cup B: 30

12. Maximum profit Rs. 1300, No. of belts of type A = 200 No. of belts of type B = 600.

13. No. of first class ticket = 40, No. of second class ticket = 160.

14. Food A: 5 units, Food B: 30 units

    A diet containing all the nutrients in appropriate quantity is called balanced diet. It is important to have all the nutrients in our diet to keep the body healthy.

15. Maximum interest is Rs. 1160 at (2000, 10000)

CHAPTER 13

PROBABILITY

POINTS TO REMEMBER

- **Conditional Probability:** If A and B are two events associated with any random experiment, then P(A/B) represents the probability of occurrence of event A knowing that event B has already occurred.

  $$P\ (A/B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

  P(B) ≠ 0, means that the event should not be impossible.

  P(A ∩ B) = P(A and B) = P(B) × P(A/B)

- Similarly P(A ∩ B ∩ C) = = P(A) × P(B/A) × P(C/AB)

  P (A/S) = P(A), P(A/A)= 1, P(S/A)= 1, P(A^1/B)= 1-P (A/B)

- **Multiplication Theorem on Probability:** If the event A and B are associated with any random experiment and the occurrence of one depends on the other, then

  P(A ∩ B) = P(A) × P(B/A) where P(A) ≠ 0

- When the occurrence of one does not depend on the other then these event are said to be independent events.

  Here P(A/B) = P(A) and P(B/A) = P(B)

  P(A ∩ B) = P(A) × P(B)

- Theorem on total probability: If $E_1, E_2, E_3 \ldots., E_n$ be a partition of sample space and $E_1, E_2 \ldots., E_n$ all have non-zero probability. A be any event associated with sample space S, which occurs with $E_1, or\ E_2, \ldots., or\ E_n$, then

  P(A) = P(E_1) . P(A/E_1) + P(E_2) . P(A/E_2) + ... + P(E_n) . P(A/E_n)

If A & B are independent then (i) A ∩ B^c, (ii) $A^c \cap B$ & (iii) $A^c \cap b^c$ are also independent.

- **Bayes' theorem :** Let S be the sample space and $E_1, E_2 \ldots., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with $E_1, or\ E_{2\ or}\ \ldots., E_n$, then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=0}^{n} P(E_i)P(A/E_i)}$$

- **Random variable :** It is real valued function whose domain is the sample space of random experiment.

- **Probability distribution :** It is a system of number of random variable (X), such that

X:	X_1	X_2	$X_{3\ldots.}$	X_n
P(X):	$P(X_{1)}$	$P(X_2)$	$P(X_{3)\ldots.}$	$P(X_{n)}$

Where P(x_i) > 0 and $\sum_{i=0}^{n} P(x_i) = 1$

- Mean or expectation of a random variables (X) is donated by E(X)

$$E(X) = \mu = \sum_{i=0}^{n} x_{i\ P(x_i)}$$

- Variance of X denoted by var(X) or σ_{x^2} and

$$Var(X) = \sigma_{x^2} = \sum_{i=0}^{n} (x_i - \mu)^2 P(X_i) = \sum_{i=0}^{n} x_i^2 P(x_i) - \mu^2$$

- The non-negative number $\sigma_x = \sqrt{var(X)}$ is called standard deviation of random variable X.

- **Bernoulli Trials:** Trials of random experiment are called Bernoulli trails if:

(xi) Number of trials is finite.

(xii) Trials are independent.

(xiii) Each trial has exactly two outcomes-either success or failure.

(xiv) Probability of success remain same in each trail.

- **Binomial distribution:**

P(X = r) = n_{c_r} , q^{n-r} p^r, where r= 0, 1, 2, …n

P = Probability of Success

q = Probability of Failure

n = total number of trails

r = value of random variables.

Very Short Answer Type Question (1 Mark)

1. Find P (A/B) if P(A)= 0.4, P(B)= 0.8 and P (B/A)= 0.6

2. Find P (A ∩ B) if A and B are two events such that P(A) = 0.5, P(B) = 0.6 and P (A ∪ B) = 0.8

3. A soldier fires three bullets on enemy: The probability that the enemy will be killed by one bullet is 0.7. What is the probability that the enemy is still alive?

4. If A and B are two events such that P(A)≠0, then find P(B/A) if (i) A is a subset of B; (ii) A∩B=∅

5. If P(A)= $\frac{1}{2}$, P(B)= $\frac{7}{12}$ and P (not A or not B)= $\frac{1}{4}$. State whether A and B are independent.

6. Three coins are tossed once. Find the probability of getting at least one head.

7. The probability that a student is not a swimmer is $\frac{1}{5}$. Find the probability that out of 5 students, 4 are swimmers.

8. Find P(A/B), if P(B) = 0.5 and P(A $\cap B$) = 0.32

9. A random variables X has the following probability distribution.

X	0	1	2	3	4	5
P(X)	$\frac{1}{15}$	k	$\frac{15k-2}{15}$	k	$\frac{15k-1}{15}$	$\frac{1}{15}$

Find the value of k.

10. (a) If P(A)= $\frac{1}{2}$, P(AUB)= $\frac{3}{5}$. And P(B)=q,find the value of q if A and B are (i) Mutually exclusive, (ii) Independent events

(b)If P(A)= $\frac{3}{10}$, P(B)= $\frac{2}{5}$ and P(AUB)= $\frac{3}{5}$ then find P(B/A)+P(A/B)

(c) If E and F are independent events such that P(E)= $\frac{3}{5}$ and P(F)= $\frac{4}{9}$, then find P$(\overline{E} \cap \overline{F})$.

Short Answer Type Questions (4 Marks)

11. A problem in mathematics is given to three students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem is solved?

12. A die is rolled. If the outcome is an even number, what is the probability that it is a prime?

13. If A and B are two events such that

P(A)= $\frac{1}{4}, P(B) = \frac{1}{2}$ and P (A $\cap B$)= $\frac{1}{8}$. Find P (not A and not B).

14. Two aeroplanes X and Y bomb a target in succession. There probabilities to hit correctly are 0.3 and 0.2 respectively. The second

plane will bomb only if first miss the target. Find the probability that target is hit by Y plane.

15. A can hit a target 4 times in 5 shots B three times in 4 shots and C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit?

16. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.

17. A and B throw a die alternatively till one of them throws a ‘6’ and wins the game. Find their respective probabilities of winning, if A starts the game.

18. A pair of die is rolled six times. Find the probability that a third sum of 7 is observed in sixth throw.

19. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

20. Two cards are drawn from a pack of well shuffled 52 cards one by one with replacement. Getting an ace or a spade is considered a success. Find the probability distribution for the number of successes.

21. In a game, a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.

22. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability that the selected person is male assuming that there are 60% males and 40% females?

23. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds?

24. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

25. If A and B are two independent events such that P$\left(\overline{A} \cap B\right) = \frac{2}{5}$) and P$\left(A \cap \overline{B}\right) = \frac{1}{6}$ then find P(A) and P(B).

Long Answer Type Questions (6 Marks)

26. In a hurdle race, a player has to cross 8hurdles. The probability that he will clear a hurdle is $\frac{4}{5}$, what is the probability that he will knock down in fewer than 2 hurdles?

27. Bag A contains 4 red, 3 white and 2 black balls. Bag B contains 3 res, 2 white and 3 black balls. One ball is transferred from bag A to bag B and then a ball is drawn from bag B. The ball so drawn is found to be red. Find the probability that the transferred ball is black.

28. If a fair coin is tossed 10 times, find the probability of getting.

(i) Exactly six heads,

(ii) at least six heads,

(iii) at most six heads.

29. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$ .The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$ if he comes by train, bus and scooter respectively but if comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

30. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six. What is the importance of “Always Speak the Truth”?

31. An insurance company insured 2000 scoter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver? Which mode of transport would you suggest to a student and why?

32. Three cards from a pack of 52 cards are lost. One card is drawn from the remaining cards. If drawn card is heart, find the probability that the lost cards were all hearts.

33. A box X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

34. In answering a question on a multiple choice, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be incorrect with probability $\frac{1}{4}$. What is the probability that the student knows the answer, given that he answered correctly?

35. Suppose a girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head. What is the probability that she throws 1, 2, 3 or 4 with the die?

36. In a bolt factory machines, A, B and C manufacture bolts in the ratio 6:3:1. 2%, 5% and 10% of the bolts produced by them respectively are defective. A bolt is picked up at random from the product and is found to be defective. What is the probability that it has been manufactured by machine A?

37. Two urns A and B contain 6 black and 4 white, 4 black and 6 white balls respectively. Two balls are drawn from one of the urns. If both the balls drawn are white, find the probability that the balls are drawn from urn B.

38. Two cards are drawn from a well shuffled pack of 52 cards. Find the mean and variance for the number of face cards obtained.

39. A letter is known to have come from TATA NAGAR or from CALCUTTA on the envelope first two consecutive letters 'TA' are visible. What is the probability that the letter come from TATA NAGAR?

40. Two groups are competing for the position on the Board of Directors of a corporation. The probabilities that first and the second group will win are 0.6 and 0.4 respectively. Further if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

41. Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6, 7. Let X denotes the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X.

42. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white?

43. Find the probability distribution of the number of doublets in four throws of a pair of dice. Also find the mean and S.D. of his distribution.

44. Three critics review a book. Odds in favour of the book are 5:2, 4:3 and 3:4 respectively for the three critics. Find the probability that the majority are in favour of the book.

45. A box contains 2 Black, 4 White and 3 Red balls. One by one all balls are drawn without replacement and arranged in sequence of drawing. Find the probability that the drawn balls are in sequence of BBWWWRRR.

46. A bag contains 3 White, 3Black and 2 Red balls. 3 balls are successively drawn without replacement. Find the probability that third ball is red.

47. If three squares are chosen at random on a chess board. Show that the chance that they should be in a diagonal line is $\frac{7}{744}$

Answers

1. 0.3
2. $\frac{3}{10}$
3. $(0.3)^3$
4. $(i)\ 1, (ii) =\ 0$
5. No
6. $\frac{7}{8}$
7. $\left(\frac{4}{5}\right)^4$
8. $\frac{16}{25}$
9. $k = \frac{4}{15}$
10. (a) (i) $\frac{1}{10}$, (ii) $\frac{1}{5}$; (b) $\frac{7}{12}$; (c) $\frac{2}{9}$
11. $\frac{3}{4}$
12. $\frac{1}{3}$
13. $\frac{3}{8}$
14. $\frac{8}{25}$

15. $\frac{5}{6}$

16. $\frac{5}{9}$

17. $\frac{6}{11}, \frac{5}{11}$

18. $\frac{625}{3888}$

19. 0.3678 or $11_{C_5}\ (0.4)^5\ (0.6)^5$

20.

X	0	1	2
P(X)	81/169	72/169	16/169

21. $-\frac{91}{54}$

22. $\frac{3}{4}$

23. $\frac{1}{17}$

24. $1-\left(\frac{9}{10}\right)^{10}$

25. $P(A)=\frac{1}{5}, P(B)=\frac{1}{6}$ or $P(A)=5/6,$ P(B)= 4/5

26. $\frac{12}{5}\left(\frac{4}{5}\right)^{7}$

27. $\frac{6}{31}$

28. $(i)\ \frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

29. $\frac{1}{2}$

30. $\frac{3}{8}$ by speaking truth, integrity of character develops.

31. $\frac{1}{52}$ Cycle should be promoted as it is good for (I) Health (II) No pollution (III) Saves energy

32. $\frac{10}{49}$

33. $\frac{25}{52}$

34. $\frac{12}{13}$

35. $\frac{8}{11}$

36. $\frac{12}{37}$

37. $\frac{5}{7}$

38. Mean $=\frac{6}{13}$ Variance$=\frac{974}{2873}$

39. $\frac{7}{11}$

40. $\frac{2}{9}$

41. Mean $=\frac{17}{3}$ Variance$=\frac{14}{9}$

42. $\frac{1}{2}$

43. Mean = 2/3 S.D.= $\frac{\sqrt{5}}{3}$

44. $\frac{209}{343}$

45. $\frac{1}{1260}$

46. $\frac{1}{4}$

MATHEMATICS 2016

Time allowed: 3 hours **Maximum Marks: 100**

General Instructions:

(i) All questions are compulsory.

(ii) Please check that this question paper contains **26** questions.

(iii) Questions **1-6** in section A are very short answer type questions carrying 1 **mark** each.

(iv) Question **7-19** in section B are long answer **I** type questions carrying **4 marks** each.

(v) Question **20-26** in Section C are long answer **II** type questions carrying **6 marks** each.

(vi) Please write down the serial number of the question before attempting it.

SECTION – A

Question numbers 1 to 6 carry 1 mark each.

1. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+sin\theta & 1 \\ 1 & 1 & 1+cos\theta \end{vmatrix}$

2. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A-I)^3 + (A+I)^3 - 7A$.

3. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is given to be symmetric, find values of a and b.

4. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}\ and 2\vec{a} + \vec{b}$ externally in the ratio 2:1.

5. The two vectors $\hat{j} + \hat{k}\ and\ 3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. find the length of the median through A.

6. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$.

SECTION – B

Question numbers 7 to 19 carry 4 marks each.

7. Prove that:

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

OR

Solve for x:

$$2\tan^{-1}(\cos x) = \tan^{-1}(2cosec\ x)$$

8. The monthly incomes of Aryan and Babban are in the ratio 3:4 and their monthly expenditures are in the ratio 5:7. If each saves ₹ 15,000 per month, find their monthly incomes using matrix method. This problem reflects which value?

9. If $x = a\sin 2t\,(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$.

OR

If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

10. Find the value of p and q, for which

$$f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x} & \text{if } x < \frac{\pi}{2} \\ p & \text{if } x = \pi/2 \\ \frac{q(1-\sin x)}{(\pi-2x)^2} & \text{if } x > \pi/2 \end{cases}$$

Is continuous at $x = \pi/2$.

11. Show that the equation of normal at any point t on the curve $x = 3\cos t - cos^3 t$ and $y = 3\sin t - \ sin^3 t$ is

$4(y\ cos^3 t - x\ sin^3 t) = 3\sin 4t$.

12. Find:

$$\int \frac{(3\ \sin\theta - 2)\cos\theta}{5 - cos^2\theta - 4\sin\theta}$$

OR

Evaluate:

$$\int_0^\pi e^{2x}.\sin\left(\frac{\pi}{4} + x\right) dx$$

13. Find:

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx.$$

14. Evaluate:

$$\int_{-1}^{2} |x^3 - x| dx.$$

15. Find the particular solution of the differential equation

$(1 - y^2)(1 + \log x)dx + 2xy\ dy = 0$, given that y = 0 when x = 1.

16. Find the general solution of the following differential equation:

$$(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$$

17. Show that the vector $\vec{a}, \vec{b}\ and\ \vec{c}$ are coplanar if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.

18. Find the vector and Cartesian equations of the line through the point (1, 2, -4) and perpendicular to the two lines.

$\vec{r} = (8\hat{\imath} - 19\hat{\jmath} + 10\hat{k}) + \lambda(3\hat{\imath} - 16\hat{\jmath} + 7\hat{k})$ and

$\vec{r} = (15\hat{\imath} + 29\hat{\jmath} + 5\hat{k}) + \mu(3\hat{\imath} + 8\hat{\jmath} - 5\hat{k})$.

19. Three persons A, B and C apply for job of Manager in a Private Company. Chances of their selection (A, B, and C) are in the ratio 1:2:4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the change does not take place, find the probability that it is due to the appointment of C.

OR

A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins.

SECTION – C

Question numbers 20 to 26 carry 6 marks each.

20. Let f : $N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that f : $N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

21. Prove that $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$ is divisible by (x + y + z), and hence find the quotient.

OR

Using elementary transformations, find the inverse of the matrix A= $\begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and use it to solve the following system of linear equations:

$8x + 4y + 3z = 19$

$2x + y + z = 5$

$x + 2y + 2z = 7$

22. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere.

OR

Find the intervals in which $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$, is strictly increasing or strictly decreasing.

23. Using integration find the area of the region

$\{(x, y): x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}$

24. Find the coordinate of the point P where the line through A(3, -4, -5) and B(2, -3, 1) crosses the plane passing through three points L(2, 2, 1), M(3, 0, 1) and N(4, -1, 0). Also, find the ratio in which P divides the line segment AB.

25. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also find mean and variance of the distribution.

26. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at ₹ 7 profit and that of B at a point of 4. Find the production level per day for maximum profit graphically.

SOLUTION OF MATHEMATICS: 2016

Section – A

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix} = 1(1+\cos\theta+\sin\theta+\sin\theta\cos\theta-1)$

$$-1(1+\cos\theta-1)+1(1-1-\sin\theta)$$

$$= \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$\therefore MAX\ value = \frac{1}{2}$

2. $(A-I)^3+(A+I)^3-7A$

$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$

$= IA - I - 3I.I + 3AI + IA + I + 3I.I + 3AI - 7A$

$$= A - I - 3I + 3A + A + A + I + 3I + 3A - 7A$$

$$= 8A - 7A = A$$

3. 2b = 3 and 3a= -2 ⇒ $b = \frac{3}{2}, a = -2/3$

4. P.v of reqd point $= \frac{2(2\vec{a}+\vec{b})-1(\vec{a}-2\vec{b})}{2-1}$

$$= 3\vec{a} + 4\vec{b}$$

5. Length of median through A $= \frac{1}{2}\left|\overrightarrow{AB}+\overrightarrow{AC}\right|$

$$= \frac{1}{2}\left|3\hat{\imath}+5\hat{k}\right| = \frac{\sqrt{34}}{2}$$

6. Reqd. Plane $\vec{r}\,.\frac{[2\hat{\imath}-3\hat{\jmath}+6\hat{k}]}{7} = 5$

Or $\vec{r}\,.[2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}] = 35$

SECTION – B

7. LHS $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$

$$= \tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}.\frac{1}{7}}\right] + \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}.\frac{1}{8}}\right]$$

$$= \tan^{-1}\left[\frac{12}{34}\right] + \tan^{-1}\left[\frac{11}{23}\right]$$

$$\tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right]$$

$$\tan^{-1}\left[\frac{138+187}{391-66}\right] = \tan^{-1}\left[\frac{325}{325}\right]$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = RHS$$

OR

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\ cosec\ x)$$

$$= \tan^{-1}\left[\frac{2\ \cos x}{1-cos^2 x}\right] = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$= \frac{2\cos x}{sin^2 x} = \frac{2}{\sin x} = \cot x = 1 \quad \Rightarrow x = \frac{\pi}{4}$$

8. Let monthly income of Aryan and Babban be Rs 3x and 4x respectively

Let monthly expenditures of Aryan and Babban be 5y and 7y respectively

$\therefore 3x - 5y = 15000$

$4x - 7y = 15000$

Now, let A = $\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$\therefore A\mathrm{X} = B, \Rightarrow \quad X = A^{-1}B$

$A^{-1} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$

$\mathrm{X} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$

$x = 30{,}000, \quad y = 15000$

$\therefore$ incomes are Rs 90,000 and Rs 1,20,000

9. $x = a \sin 2t\,(1 + \cos 2t),\ y = b\ \cos 2t\ (1 - \cos 2t)$

$\frac{dx}{dt} = a[\sin 2t\,(-2\ \sin 2t) + 2\cos 2t\,(1 +\ \cos 2t)]$

$= a[2\ \cos 2t + 2\cos 4t]$

$\frac{dy}{dt} = b[\cos 2t\,(2\ \sin 2t) - 2\sin 2t\,(1 -\ \cos 2t)]$

$= b[2\sin 4t - 2\sin 2t + \sin 4t]$

$= b[2\sin 4t - 2\sin 2t]$

$$\frac{dy}{dx} = \frac{2b\ [\sin 4t - \sin 2t]}{2a\ [\cos 4t + \cos 2t]}$$

$$\left(\frac{dy}{dx}\right) t = \frac{\pi}{4} = \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}$$

$$\left(\frac{dy}{dx}\right)\ t = \frac{\pi}{3} = \frac{b}{a}\left[\frac{-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}}{-\frac{1}{2}-\frac{1}{2}}\right] = \frac{b}{a}\ \left(\frac{-\sqrt{3}}{-1}\right) = \frac{b\sqrt{3}}{a}$$

OR

$$y = x^x \Rightarrow \log y = \log x$$

$$\frac{dy}{dx} = x^x(1 + \log x)$$

$$\frac{d^2y}{dx^2} = x^x \times \frac{1}{x} + (1 + \log x).\, x^x(1 + \log x)$$

$$= \frac{x^x}{x} + x^x(1 + \log x)^2$$

LHS $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}$

$$= \frac{x^x}{x} + x^x(1 + \log x)^2 - \frac{1}{x^x}\,.(x^x)^2(1 + log\ x)^2 - \frac{x^x}{x}$$

$$= x^{x-1} + x^x(1 + log\ x)^2 - x^x(1 + \log x)^2 - x^{x-1}$$

= 0 = RHS

10. $\underset{x \to \frac{\pi}{2}}{\text{lt}} - f(x) = \underset{h \to 0}{\text{lt}} \frac{1 - sin^3\left(\frac{\pi}{2} - h\right)}{3cos^2\left(\frac{\pi}{2} - h\right)}$

$$= \underset{h \to 0}{\text{lt}} \frac{1 - cos^3 h}{3sin^2 h}$$

$$= \underset{h \to 0}{\text{lt}} \frac{(1 - \cos h)(1 + cos^2 h + \cos h)}{3(1 - \cos h)(1 + \cos h)} = \frac{(1 + 1 + 1)}{3(1 + 1)} = \frac{1}{2}$$

$$\underset{x \to \frac{\pi}{2}}{\text{lt}} + f(x) = \underset{h \to 0}{\text{lt}} \frac{q\left(1 - sin\left(\frac{\pi}{2} + h\right)\right)}{\left(\pi - 2\left(\frac{\pi}{2} + h\right)\right)^2}$$

$$\underset{h \to 0}{\text{lt}} \frac{q(1 - \cos h)}{(-2h)^2} = \underset{h \to 0}{\text{lt}} \frac{q.2\, sin^2\, h/2}{4h^2}$$

$$\underset{h \to 0}{\text{lt}} \frac{q}{2} \times \left(\frac{\sin h/2}{h/2}\right)^2 \times \frac{1}{4} = \frac{q}{8}$$

$$\text{f}\left(\frac{\pi}{2}\right) = p$$

for function to be continuous at x=$\frac{\pi}{2}$, we must have

$$\underset{x \to \frac{\pi}{2}}{\text{lt}} - f(x) = \underset{x \to \frac{\pi}{2}}{\text{lt}} + f(x) = f(\frac{\pi}{2})$$

i.e. $\frac{1}{2} = \frac{q}{8} = p \Rightarrow$ $\boxed{p = \frac{1}{2}\ \&\ q = 4}$

11. $x = 3\cos t - cos^3 t \Rightarrow \frac{dx}{dt} = -3\sin t + 3\, cos^2 t + \sin t$

$$= -3\sin t\,(1 - cos^2 t)$$

$$= -3sin^3 t$$

$$y = 3\sin t - sin^3 t \Rightarrow \frac{dy}{dt} = 3\cos t - 3sin^2 t\ \cos t$$

$$= 3\cos t\,(1 - sin^2 t)$$

$$= 3\,cos^3 t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx} = -cot^3 t$$

Equation of normal at any point t

$$y - 3\sin t + sin^3 t = \frac{sin^3 t}{cos^3 t}\,(x - 3\cos t + cos^3 t)$$

$$y\,cos^3 t - 3\sin t\,cos^3 t + sin^3 t cod^3 t = x sin^3 t - 3sin^3 t\cos t + sin^3 t cos^3 t$$

$$y cos^3 t - x sin^3 t = 3\sin t\,cos^3 t - 3sin^3 t\cos t$$

$$= 3\sin t\,cost\,[cos^2 t - sin^2 t]$$

$$= \frac{3}{2}\sin 2t\,.\cos 2t$$

$$= y\,cos^3 t - x sin^3 t = \frac{3}{4}\sin 4t$$

$$\boxed{= 4y(cos^3 t - x sin^3 t) = 3\sin 4t}$$

12. $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - cos^2\theta - 4\sin\theta}\ d\theta$

$I = \int \frac{(3\sin\theta - 2)\cos\theta}{4 + sin^2\theta - 4\sin\theta}\ d\theta$ put $\sin\theta = t$

Cos $\theta\ d\theta = \alpha t$

$I = \int \frac{3t - 2}{t^2 - 4t + 4}\ dt$

$I = \int \frac{3t-2}{(t-2)^2} \quad dt$ put t – 2 = y ⇒dt=dy

$I = \int \frac{3(y+2)-2}{y^2} \quad dy$

$I = \int \frac{3y+4}{y^2} \quad dy$ ⇒I = $3 \left]\frac{1}{y} dy + 4\right] \frac{1}{y^2} dy$

$$I = 3\log|y| - \frac{4}{y} + c$$

$$I = 3\log|t-2| - \frac{4}{t-2} + c$$

$$I = 3\log|\sin\theta - 2| - \frac{4}{(\sin\theta - 2)} + c$$

OR

$$I = \int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx$$

Consider $I_1 = \int e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx$

$$I_1 = \sin\left(\frac{\pi}{4} + x\right).\left(\frac{e^{2x}}{2}\right) - \int \cos\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} \, dx$$

$$I_1 = \frac{e^{2x}\sin\left(\frac{\pi}{4}+x\right)}{2} - \frac{1}{2}\left[cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}\right] - \int - \sin\left(\frac{\pi}{4} + x\right).\frac{e^{2x}}{2} \, dx$$

$$I_1 = \frac{1}{2}.e^{2x}\sin\left(\frac{\pi}{4} + x\right) - \frac{1}{4}e^{2x}\cos\left(\frac{\pi}{4} + x\right) - \frac{1}{4}I_1$$

$$\frac{5}{4}I_1 = \frac{e^{2x}}{4}\left[2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right]$$

$$I_1 = \frac{e^{2x}}{5}\left[2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right]$$

$$I = \left[\frac{e^{2x}}{5}\left[2\sin\left(\frac{\pi}{4}+x\right) - cos\left(\frac{\pi}{4}+x\right)\right]\right]$$

$$I = \left[\frac{e^{2\pi}}{5}\left[2\sin\left(\frac{\pi}{4}+\pi\right) - cos\left(\frac{\pi}{4}+\pi\right)\right] - \frac{e^{0}}{5}\left[2\sin\frac{\pi}{4} - cos\frac{\pi}{4}\right]\right]$$

$$I = \frac{e^{2\pi}}{5}\left[-\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right] - \frac{1}{5}\left[\frac{2}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left(\frac{1}{\sqrt{2}}-\sqrt{2}\right) - \left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\right]$$

13. $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

$I = \int \frac{\sqrt{x}}{\sqrt{a^3-\left(x^{3/2}\right)^2}}dx$ put $x^{3/2} = t \Rightarrow \frac{3}{2}.x^{1/2}dx = dt$

$$I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$$

$$I = \frac{2}{3}sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c$$

$$I = \frac{2}{3}sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c$$

14. $\int_{-1}^{2}|x^3 - x|dx$ $\qquad x(x-1)(x+1)$

$x(x^2-1) = 0$

X= -1, 0, 1

$$I = \int_{-1}^{0}|x^3 - x|dx + \int_{0}^{1}|x^3 - x|dx + \int_{1}^{2}|x^3 - x|\,dx$$

$$I = \int_{-1}^{0}(x^3 - x)dx + \int_{0}^{1}(x - x^3)dx + \int_{1}^{2}(x^3 - x)\,dx$$

$$= \frac{1}{4}[x^4]_{-1}^{0} - \frac{1}{2}[x^2]_{-1}^{0} + \frac{1}{2}[x^2]_{0}^{1} - \frac{1}{4}[x^4]_{0}^{1} + \frac{1}{4}[x^4]_{1}^{2} - \frac{1}{2}[x^2]_{1}^{2}$$

$$= \frac{1}{4}[0-1] - \frac{1}{2}[0-1] + \frac{1}{2}[1-0] - \frac{1}{4}[1-0] + \frac{1}{4}[15] - \frac{1}{2}[3]$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{15}{4} - \frac{3}{2}$$

$$= \frac{15}{4} - 1 = \frac{11}{4}$$

15. $(1-y^2)(1+\log x)dx + 2xy\,dy = 0$ $\qquad$ y = 0 when x = 1

$$= (1-y^2)(1+\log x)dx = -2xy\,dy$$

$$= \int\left(\frac{1+\log x}{x}\right)dx = -\int\frac{2y}{1-y^2}dy$$

$$= \int\frac{1}{x}dx + \int\frac{\log x}{x}dx = \int\frac{2\,y}{y^2-1}dy$$

$$= \log|x| + \frac{(\log x)^2}{2} = \log|y^2 - 1| + c$$

$$\log|x| + \frac{1}{2}(\log x)^2 = \log|y^2 - 1| + c$$

As y = 0 when x=1, we get

$$\log|1| + \frac{1}{2}(\log(1))^2 = \log|0 - 1| + c$$

$$\Rightarrow 0 + 0 = 0 + c \Rightarrow c = 0$$

∴ particular solution is

$$= \log|x| + \frac{1}{2}(\log|x|)^2 = \log|y^2 - 1|$$

$$= \log|x^2| - \log|y^2 - 1| + \frac{1}{2}(\log x)^2 = 0$$

$$= \log\left|\frac{x^2}{y^2 - 1}\right| + \frac{(\log x)^2}{2} = 0$$

16. $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

$$= \left(e^{\tan^{-1} y} - x\right)\frac{dy}{dx} = (1 + y^2)$$

$$= \frac{dx}{dy} = \frac{e^{\tan^{-1} y} - x}{1 + y^2}$$

$$= \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$I.f = e^{\int P dx} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Gen. Solution:

$$x(I.F) = \int x\,.(I.F)dy + c$$

$$= xe^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2}.e^{\tan^{-1} y}\,dy + c$$

$$= \int \frac{e^{2\tan^{-1} y}}{1+y^2}dy \qquad \text{put } \tan^{-1} y = t$$

$$= \frac{1}{1+y^2}dy = dt$$

$$= xe^{\tan^{-1} y} = \int e^{2t}dt + c$$

$$= \frac{e^{2t}}{2} + c$$

$$= 2xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$$

17. Consider $(\vec{a}+\vec{b}).\left[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right]$

$$= (\vec{a}+\vec{b}).\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{c}\right] \qquad [\vec{c}\times\vec{c}=0]$$

$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]+0+0+0+0+[\vec{b}\,\vec{c}\,\vec{a}]$$

$$= 2[\vec{a}\,\vec{b}\,\vec{c}]$$

$i.e.\left[(\vec{a}+\vec{b})\ (\vec{b}+\vec{c})\ (\vec{c}+\vec{a})\right] = 2\ [\vec{a}\,\vec{b}\,\vec{c}]$________(1)

$since\ \vec{a}+\vec{b}, \vec{b}+\vec{c}$ & $\vec{c}+\vec{a}$ are coplanar

$= \left[(\vec{a}+\vec{b})\ (\vec{b}+\vec{c})\ (\vec{c}+\vec{a})\right] = 0$

$= 2\ [\vec{a}\ \vec{b}\ \vec{c}] = 0$ ____________[using (1)]

$= \ [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\therefore \vec{a}\ \vec{b}\ and\ \vec{c}$ are coplanar

18. $\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 24\hat{i} + 36\hat{j} + 72\hat{k}$

Vector eqn. $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(24\hat{i} + 36\hat{j} + 72\hat{k})$

Cartesian eqn. $\frac{x-1}{24} = \frac{y-2}{36} = \frac{z+4}{72}$

Or $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

19. Let event A_1 = Person A is selected for the job

A_2 = Person B is selected for the job

A_3 = Person C is selected for the job

B = change does not take place.

$P(A_1) = \frac{1}{7} \quad P(A_2) = \frac{2}{7} \quad P(A_3) = \frac{4}{7}$

$P\left(\frac{B}{A_1}\right) = 0.2 \qquad P\left(\frac{B}{A_2}\right) = 0.5 \qquad P\left(\frac{B}{A_3}\right) = 0.7$

$P\left(\frac{A_3}{B}\right) = \frac{P(A_3)\ P(B/A_3)}{P(A_1)\ P\left(\frac{B}{A_1}\right) + P(A_2)\ P\left(\frac{B}{A_2}\right) + P(A_3)\ P\left(\frac{B}{A_3}\right)}$

$$= \frac{\frac{4}{7}\times 0.7}{\frac{1}{7}\times 0.2+\frac{2}{7}\times 0.5+\frac{4}{7}\times 0.7}$$

$$= \frac{2.8}{0.2+1.0+2.8} \quad = \frac{2.8}{4.0} \quad = \frac{28}{40} \quad \frac{7}{10}$$

OR

Let p = prob. Of getting a total of $7\{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\}$

Q = prob. Of getting a total of $10\{(4,6)(5,5)(6,4)\}$

P (A wins) $= p + \overline{p}\,\overline{q}p + \overline{p}\,\overline{q}\,\overline{p}\,\overline{q}\,p + \cdots$

$$= \frac{6}{36} + \frac{30}{36} \times \frac{33}{36} \times \frac{6}{36} + \left(\frac{30}{36} \times \frac{33}{36}\right)^2 \frac{6}{36} + \cdots$$

$$= \frac{1}{6}\left[1 + \frac{5}{6} \times \frac{11}{12} + \left(\frac{5}{6} \times \frac{11}{12}\right)^2 + \cdots\right]$$

$$= \frac{1}{6}\left[\frac{1}{1-\frac{55}{72}}\right] \quad = \frac{1}{6} \times \frac{72}{17} \quad = \frac{12}{17}$$

P (A wins)+ P(B wins)=1 ⇒ P(B wins)= $\frac{5}{17}$

20. Consider $x_1, x_2\ E\ N$

Let $f(x_1) = f(x_2)$

$= 9{x_1}^2 + 6x_1 - 5 = 9{x_2}^2 + 6x_2 - 5$

$= (3x_1 + 1)^2 - 6 = (3x_2 + 1)^2 - 6$

$= 3x_1 + 1 = 3x_2 + 1 = \ 3x_1 = 3x_2 = x_1 = x_2$

∴ f is one – one

As $f: N \to S$ where S is Δ range of f

$\therefore$ f is onto (given)

$\therefore$ f is a bijective function

Hence f is invertible

Let $y = f(x)$

$$y = 9x^2 + 6x - 5$$

$$y = (3x+1)^2 - 6$$

$$x = \frac{\sqrt{y+6}-1}{3}$$

Now $f^{-1}: S \to N$

$$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

$$f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = \frac{7-1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = \frac{12}{3} = 4$$

21. Let $\Delta = \begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$

$c_1 \to c_1 + c_2 + c_3$

$$\begin{vmatrix} xy + yz + zx - x^2 - y^2 - z^2 & zx - y^2 & xy - z^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & xy - z^2 & yz - x^2 \\ xy + yz + 2x - x^2 - y^2 - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1, \;\; R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} xy+yz+zx-x^2-y^2-z^2 & zx-y^2 & xy-z^2 \\ 0 & x(y-z)+(y^2-z^2) & y(z-x)+(z^2-x^2) \\ 0 & z(y-x)+(y^2-x^2) & x(z-y)+(z^2-y^2) \end{vmatrix}$$

$$= xy+yz+zx-x^2-y^2-z^2 \begin{vmatrix} (y-z)(x+y+z) & (z-x)(y+z+x) \\ (y-x)(z+y+x) & (z-y)(x+z+y) \end{vmatrix}$$

$$\Delta = (xy+yz+zx-x^2-y^2-z^2)(x+y+z)^2 \begin{vmatrix} y-z & z-x \\ y-x & z-y \end{vmatrix}$$

$= (xy+yz+zx-x^2-y^2-z^2)(x+y+z)^2 \quad [yz-y^2+yz-z^2-yz+yx+xz-x^2]$

$\Delta = (xy+yz+zx-x^2-y^2-z^2)^2 \ (x+y+z)^2$

$hence$ Δ is divisible by (x+y+z) and quotient is (x+y+z) $(xy+yz+zx-x^2-y^2-z^2)^2$

OR

As A = IA

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 4R_2, \quad R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 1 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 1 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3, \quad R_2 \to R_2 - R_3$

$$\begin{bmatrix} -3 & 0 & 0 \\ 5 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} A$$

$R_1 \to \frac{-1}{3} R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_1, \; R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} A$$

$$\therefore \; A^{-1} = \frac{1}{3}\begin{bmatrix} 0 & 2 & -1 \\ 3 & -13 & 2 \\ -3 & 12 & 0 \end{bmatrix}$$

$8x + 4y + 3z = 19, \quad 2x + y + z = 5, \quad x + 2y + 2z = 7$

$$\text{A}=\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \qquad \text{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad A = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$\therefore AX = B$

$X = A^{-1}B$

$$\therefore X = \frac{1}{3}\begin{bmatrix} 0 & 2 & -1 \\ 3 & -13 & 2 \\ -3 & 12 & 0 \end{bmatrix}\begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

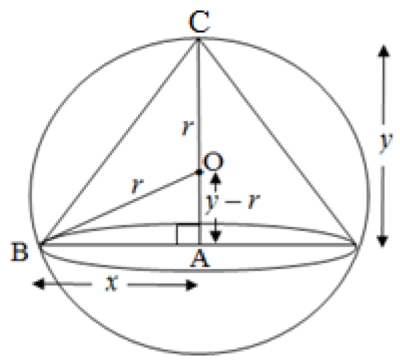
$$X = \frac{1}{3}\begin{bmatrix} 0 + 10 - 7 \\ 57 - 65 + 14 \\ -57 + 60 + 0 \end{bmatrix}$$

$X = \frac{1}{3}\begin{bmatrix}3\\6\\3\end{bmatrix}$

$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ X=1, y=2, z=1

22. Let radius and height of cone be x units and y units respectively

in right ΔOAB



$x^2 + (y-r)^2 = r^2$

$x^2 = r^2 - y^2 - r^2 + 2yr$

$x^2 = -y^2 + 2yr$

$v = \frac{1}{3}\pi r^2 y$

$v = \frac{1}{3}\pi[-y^2 + 2yr]y$

$v = \frac{1}{3}\pi[-y^3 + 2y^2 r]$

$\frac{dv}{dy} = \frac{\pi}{3}[-3y^2 + 4yr]$

$\frac{dv}{dy} = 0 \quad = \frac{\pi y}{3}[-3y + 4r] = 0$

$= \frac{y=0}{rejected}$ or $y = \frac{4r}{3}$

$\frac{d^2y}{dy^2} = \frac{\pi}{3}[-6y + 4r]$

$$\frac{d^2y}{dy^2} = \frac{\pi}{3}[-8r + 4r]\angle 0$$

$$y = \frac{4r}{3}$$

V is maximum when $y = \frac{4r}{3}$

now max. Volume of cone

$$= \frac{\pi}{3}\left[-\frac{64r^3}{27} + 2\text{x}\frac{16r^2}{9} \times r\right]$$

$$= \frac{\pi}{3}\left[-\frac{64r^3}{27} + \frac{32r^2}{9}\right]$$

$$= \frac{\pi}{3}\left[\frac{-64r^3 + 96r^3}{27}\right]$$

$$= \frac{32\pi r^3}{81}$$

$$= \frac{8}{27}\left[\frac{4}{3}\pi r^3\right]$$

$= \frac{8}{27}$ volume of sphere

OR

$f(x) = \sin 3x - \cos 3x, \qquad\qquad 0 < x < \pi$

$f'(x) = 3\cos 3x + 3\sin 3x$

For stationary points $f'(x) = 0$

$= \cos 3x + \sin 3x = 0$

$= \tan 3x = -1$

$$= x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

These stationary points divides (0,π) into four disjoint intervals

$\left(0,\frac{\pi}{4}\right)$, $(\frac{\pi}{4},\frac{7\pi}{12})$, $(\frac{7\pi}{12},\frac{11\pi}{12})$, $(\frac{11\pi}{12},\pi)$

Interval	sign of $f'(x)$	Nature of f(x)
$\left(0,\frac{\pi}{4}\right)$	+ve	↑
$\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$	-ve	↓
$\frac{11\pi}{12},\frac{11\pi}{12}$	+ve	↑
$(\frac{11\pi}{12},\pi)$	-ve	↓

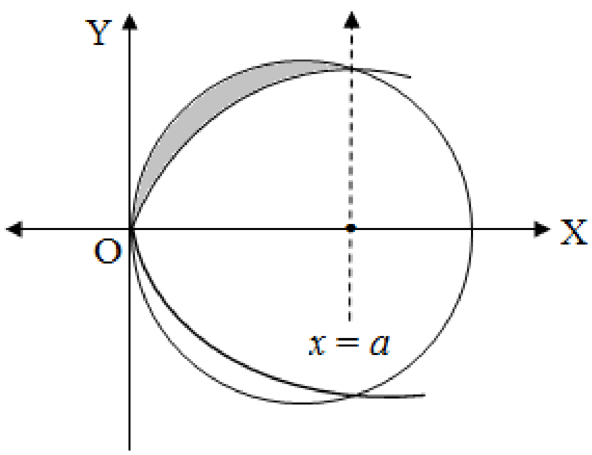
∴ f is strictly increasing on $\left(0,\frac{\pi}{4}\right) \cup (\frac{7\pi}{12}, \frac{11\pi}{12})$

F is strictly decreasing on $(\frac{\pi}{4},\frac{7\pi}{12}) \cup (\frac{11\pi}{12},\pi)$

23. $\{(x,y): x^2+y^2 \leq 2ax, y^2 \geq ax, x_1 y \geq 0\}$

Let $x^2+y^2 = 2ax$

$(x-a)^2+y^2 = a^2$



$x^2 + y^2 = 2ax, \;\; y^2 = ax$

$x^2 + ax = 2ax \;\; = x^2 - ax = 0$

$= x(x - a) = 0$

$x = 0, x = a$

Required area $= \int_0^a \sqrt{a^2 - (x-a)^2 dx}$ - $\int_o^a \sqrt{ax\,dx}$

$= \left[\frac{x-a}{2}\sqrt{a^{2-}(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a$

$-\sqrt{a}.\left[\frac{x^{3/2}}{3/2}\right]_o^a$

$= \left[\left(0 + \frac{a^2}{2}\sin^{-1}(0)\right) - \left(0 + \frac{a^2}{2}\sin^{-1}(-1)\right)\right]$

$-\frac{2}{3}\sqrt{a}\left[a^{3/2} - 0\right]$

$= \frac{\pi a^2}{4} - \frac{2a^2}{3} = \frac{3\pi a^2 - 8a^2}{12} = \frac{a^2(3\pi - 8)}{12}$ sq. units

24. Equation of line through (3, -4, -5) & (2, -3, 1) is

$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$ (let)

Equation of plane through (2, 2, 1), (3, 0, 1) & (4, -1, 0) is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$= (x-2)[2-0] - (y-2)[-1-0] + (z-1)(-3+4) = 0$

$= 2x - 4 + y - 2 + z - 1 = 0$

$= 2x + y + z = 7$

Coordinates of a general point on line are $(-\lambda + 3, \; \lambda - 4, \; 6\lambda - 5)$

For point of intersection of line & plane

$2(-\lambda+3)+(\lambda-4)+(6\lambda-5)=7$

$-2\lambda+6+\lambda-4+6\lambda-5=7$

$5\lambda-3=7$

$\lambda=2$

∴ coordinates of point P are (1, -2, 7)

Let P divides AB in ratio $\lambda : 1$ *internally*

$(1,-2,7)=\left(\frac{2\lambda+3}{\lambda+1},\frac{-3\lambda-4}{\lambda+1},\frac{\lambda-5}{\lambda+1}\right)$

$\lambda=-2$

∴ P divides segment AB in ratio 2:1 externally

25. Let X denotes the number of red balls drawn x = 4, p = $\frac{6}{9}=\frac{2}{3}$, q=$\frac{3}{9}=\frac{1}{3}$

Possible values of x are 0, 1, 2, 3, 4

$P(\mathrm{x}=0)=4c_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^4 \quad =\frac{1}{81}$

$P(\mathrm{x}=1)=4c_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^3 \quad =\frac{8}{81}$

$P(\mathrm{x}=2)=4c_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \quad =\frac{24}{81}$

$P(\mathrm{x}=3)=4c_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 \quad =\frac{32}{81}$

$P(\mathrm{x}=4) = 4c_4\left(\frac{2}{3}\right)^4 \quad = \frac{16}{81}$

X	P(X)	XP(X)	$x^2P(X)$
0	1/81	0	0
1	8/81	8/81	8/81
2	24/81	48/81	96/81
3	32/81	96/81	288/81
4	16/81	64/81	256/81
		216/81	648/81

Mean $=\sum \times P(x)$

$$\frac{216}{81} = \frac{24}{9} = \frac{8}{3}$$

Variance $=\sum \times^2 P(x) - (mean)^2$

$$= \frac{648}{81} - \frac{576}{81}$$

$$= \frac{72}{81} = \frac{8}{9}$$

26. Let no. of units of product A=x, B=y

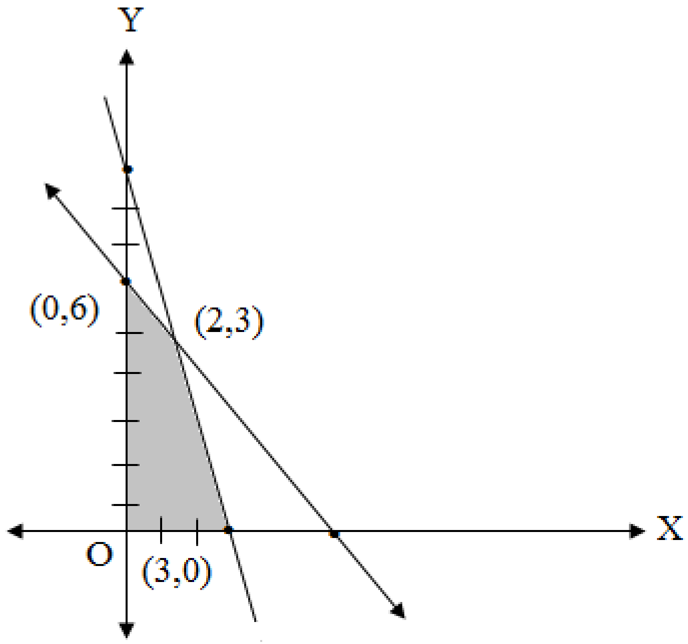
Let no. of units of product B=y

Max (z)= 7x+4y

Subject to constraints $3x + 2y \leq 12$

$$3x + y \leq 9$$

$$x \geq 0, \quad y \geq 0$$



Corner points the bounded feasible region are (0, 0), (3, 0), (2, 3) & (0, 6)

Cover points	Z=7x+4y
(0, 0)	0
(3, 0)	21
(2, 3)	26
(0, 6)	24

∴ max. Profit is Rs 26 when

no. of units of A produced per day is 2

no. of units of B produced per day is 3

MATHEMATICS

Time allowed: 3 hours **Maximum Marks: 100**

General Instructions:

(i) All questions are compulsory.

(ii) Please check that this question paper contains **26** questions.

(iii) Questions **1-6** in section A are very short answer type questions carrying **1 mark** each.

(iv) Question **7-19** in section B are long answer **I** type questions carrying **4 marks** each.

(v) Question **20-26** in Section C are long answer **II** type questions carrying **6 marks** each.

(vi) Please write down the serial number of the question before attempting it.

SECTION – A

Question number 1 to 6 carry 1 mark each.

1. If (2 1 3)$\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$= A, then write the order of matrix A.

2. If $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$= 8, write the value of x.

3. If A= $\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$ is written as A=P+Q, where P is a symmetric and Q is skew symmetric matrix, then write the matrix P.

4. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then write the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

5. If$\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2 = 400 \ and \ |\vec{a}| = 5$,then write the value of $\left|\vec{b}\right|$.

6. Write the equation of a plane which is at a distance of $5\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axes.

SECTION – B

7. Prove that: $\cot^{-1} \frac{\sqrt{1+sin\ x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \frac{x}{2}, 0 < x < \frac{\pi}{2}$

OR

Solve for x:

$$\tan^{-1}\left(\frac{x-2}{x-1}\right) + \tan^{-1}\left(\frac{x+2}{x+1}\right) = \frac{\pi}{4}$$

8. A coaching institute of English (subject) conducts classes in two batches I and II and fees for rich and poor children are different. In batch I, it has 20 poor and 5 rich children and total monthly collection is ₹ 9,000, whereas in batch II, it has 5 poor and 25 rich children and total monthly collection is ₹ 26,000. Using matrix method, find monthly fees paid by each child of two types. What values the coaching institute is including in the society?

9. Find the values of a and b, if the function f defined by

$$f(x) = \begin{cases} x^2 + 3x + a, & x \leq 1 \\ bx + 2 \quad , & x > 1 \end{cases}$$

10. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if $x \in (-1, 1)$.

OR

If $x = \sin t$ and $y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

11. Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 4by$.

12. Evaluate:

$$\int_0^{\pi} \frac{x}{1 + \sin\alpha \sin x} dx$$

13. Find: $\int (2x + 5)\sqrt{10 - 4x - 3x^2} dx$

OR

Find: $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$

14. Find: $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

15. Solve the following differential equation:

$y^2 dx + (x^2 - xy + y^2)dy = 0$

16. Solve the following differential equation:

$(\cot^{-1} y + x)dy = (1 + y^2)dx$

17. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

18. Prove that the line through A(0, -1, -1) and B(4, 5, 1) intersect the line through C(3, 9, 4) and D(-4, 4, 4)

19. A box has 20 pens of which 2 are defective. Calculate the probability that out of 5 pens drawn one by one with replacement, at most 2 are defective.

OR

Let X denote the number of colleges where you will apply after your results and P (X=x) denotes your probability of getting admission in x number of colleges. It is given that

$$P(X=x)=\begin{cases} kx & , \ if\ x = 0\ or\ 1 \\ 2\,kx & \ if\ x = 2 \\ k\,(5-x), & if\ x = 3\ or\ 4 \\ 0 & ,if\ x > 4 \end{cases}$$

Where K is a positive constant. Find the value of K. also find the probability that you will get admission in (i) exactly one college (ii) at most 2 colleges (iii) at least two colleges.

SECTION – C

Question numbers **20 to 26** carry **6** marks each.

20. If f, $g: R \to R$ be two functions defined as

$f(x) = |x| + x$ and $g(x) = |x| - x, \forall x \in R$.

Then find fog and gof. Hence find fog (-3), fog (5) and gof(-2).

21. If a, b and c are all non-zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$, then prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1 = 0.$

OR

If A=$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$, find adj.A and verify that

$A(adj.A) = (adj.A)\ A = |A|I_3$.

22. The sum of the surface areas of a cuboid with sides x, 2x, and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes.

OR

Find the equation of tangents to the curve $y = \cos(x + y), -2\pi \leq x \leq 2\pi$ that are parallel to the line $x + 2y = 0$.

23. Using integration find the area of the region bounded by the curves

$y = \sqrt{4 - x^2}, x^2 + y^2 - 4x = 0$ and the x-axis.

24. Find the equation of the plane which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ *and* $2x + y - z + 5 = 0$ and whose x-intercept is twice its z-intercept.

Hence write the vector equation of a plane passing through the point (2, 3, -1) and parallel to the plane obtained above.

25. Bag A contain 3 red and 5 black balls, while bag B contains 4 red and 4 black balls. Two balls are transferred at random from bag A to bag B and then a ball is drawn from bag B at random. If the ball drawn from bag B is found to be red, find the probability that two red balls were transferred from A to B.

26. In order to supplement daily diet, a person wishes to take to X and Y tablets. The contents (in milligrams per tablet) of iron, calcium and vitamins in X and Y are given as below:

Tablets	**Iron**	**Calcium**	**Vitamin**
X	6	3	2
Y	2	3	4

The person needs to supplement at least 18 milligrams of iron, 21 milligrams calcium and 16 milligram of vitamins. The price of each tablet of X and Y is ₹2 and ₹1 respectively. How many tablets of each type should the person take in order to satisfy the above requirement at the minimum cost? Make an LPP and solve graphically.

Answers

1. 1×1

2. $x = -2$

3. $P = \begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$

4. $\frac{-3}{2}$

5. 400

6. $x + y + z = 15$

7. (or) $x = \pm\sqrt{\frac{7}{2}}$

8. x = 200, y = 1000

9. a = 3, b = 5

10. $\frac{dy}{dx} = \frac{1}{4}$

11. $\tan^{-1}\left[\frac{3}{2}\left(\frac{a^{1/3}.b^{1/3}}{a^{2/3}+b^{2/3}}\right)\right]$ $and\ 90^\circ$

12. ..

13. $\frac{-2}{9}(10 - 4x - 3x^2)^{\frac{3}{2}} + \frac{11\sqrt{3}}{3}\left[\left(x - \frac{2}{3}\right)\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x - \frac{2}{3}\right)^2} + \right.$
$\left.\frac{17}{9}\sin^{-1}\left(\frac{3x-2}{\sqrt{34}}\right)\right] + c$

OR

$$x + \frac{1}{4\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + \frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$$

14. $-\sqrt{1-x^2}\,\sin^{-1}x + x + c$

15. $\tan^{-1}\frac{x}{y} = -\log|y| + c$

16. $x = (1-\cot^{-1}y) + c\,e^{-\cot^{-1}y}$

17. ..

18. ..

19. $\left(\frac{9}{10}\right)^3 \times \frac{24}{25}$ OR (i) k= $\frac{1}{8}$ (ii) $\frac{5}{8}$ (iii) $\frac{7}{8}$

20. $fog(-3) = 6,\quad fog(5) = 0,\quad gof(-2) = 0$

21. ..

22. min value of sum of volumes= $\left(\frac{2x^3}{3} + 36\pi x^3\right)$ cubic units

OR

$2x + 4y + 3\pi = 0,\qquad 2x + 4y - \pi = 0$

23. $\left(\frac{5\pi}{3} - \sqrt{3}\right)$ sq. Units

24. $7x + 11y + 14z = 15,$ 13x+14y+11z=0

$7x + 11y + 14z = 33,$ 13x+14y+11z=57

25. $\frac{18}{133}$

26. Min. Cost is Rs.9 at x = 0, y = 9