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BUSINESS MATHEMATICS AND STATISTICS

(For Higher Secondary Second Year)



ASSAM HIGHER SECONDARY EDUCATION COUNCIL
[ON THE BASIS OF NATIONAL CURRICULUM FRAMEWORK (NCF) 2005]

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FOREWORD

In order to impart quality education to the students of Higher Secondary level and also to cater its need of the hour Assam Higher Secondary Education Council has been revising its curriculum and syllabi time to time. It has introduced a revised curricula with effect from 2010-2011 which has its base on NCF-2005. In the context of globalization of the economy, emergence of Information Technologies and application of new technologies in production processes, the National Council of Education Research and Training has proposed a new National Curriculum Framework (NCF-2005) for grades I-XII. This framework, which addresses the emerging development issues and other social concerns, provides a basis for the state to design their curricula, syllabi, teaching learning materials etc. In its endeavour to keep the uniformity with the national level Assam Higher Secondary Education Council after due deliberation decided to develop its curricula and syllabi on the basis of NCF-2005. Accordingly textbooks have been prepared to materialize the objectives of the curricula syllabi.

Writing this book was a collective effort of a group of people. Assam Higher Secondary Education Council appreciates the hard work done by the textbook development committee responsible for this book. AHSEC welcomes comments and suggestions which will enable us to undertake further revision and refinement. From teachers and students also we would appreciate feedback about the book and its design.

Guwahati
31st May, 2022

Secretary
Assam Higher Secondary
Education Council
Bamunimaidam, Guwahati-21

CONSTITUTION OF INDIA

PREAMBLE

We, The People Of India, having solemnly resolved to constitute India into a **Sovereign Socialist Secular Democratic Republic** and to secure to all its citizen :

Justice, social, economic and political;

Liberty of thought, expression, belief, faith and worship :

Equality of status and of opportunity; and to promote among them all

Fraternity assuring the dignity of the individual and the unity and integrity of the Nation;

In Our Constituent Assembly this twenty-sixth day of November, 1949, do **Hereby Adopt, Enact And Give To Ourselves This Constitution.**

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UNIT-1**SIMPLE AND COMPOUND INTEREST, ANNUITY****SIMPLE INTEREST**

Introduction : Concept of interest on a certain sum lent to some one is not new. We have heard this word interest in many occasions. As for example, people take loan from 'Kabuliwala', merchants or any private money lender etc. Even sometimes Govt. takes loan from any financial institution or big business magnets so as to discharge certain liabilities.

Money lenders do not give loan for loan sake only. Rather they get an extra benefit from the borrower on the loan amount upto a certain period for which the loan is taken. This extra benefit is popularly known as 'interest' or a service charge. Borrower is also benefited because he can complete some unfinished work which may later give him rich dividend.

Meaning of Principal, Amount, Rate of Interest : The sum borrowed / cash at hand is known as Principal or Sum. The invested sum and interest taken together is known as Amount.

Thus, Amount = Principal + Interest

The interest is calculated on the principal at a certain rate for a certain period. This rate of interest is generally expressed in percentage eg. Rate of interest is 5% p.a. This means that interest on Rs. 100 (sum) for 1 year is Rs. 5.

- Note :**
- (i) Rate of interest is dependent on the term and risk factor.
 - (ii) In case of simple interest, principal remains the same upto the total period and interest is calculated every year on the original principal.
 - (iii) p.a. (per annum) means per year.
 - (iv) In a non-leap year, total no of days = 365. In a leap year (i.e. the year no is divisible by 4), total no of days = 366, Month of Feb has 29 days. No of weeks in a year = 52.
 - (v) Interest means simple interest unless specifically mentioned.
 - (vi) In counting the no. of days from one date to another date, one of the extreme date is taken.

eg. No of days from 15th March to 18 June = 16+30+31+18 = 95.

SOME SYMBOLS USED IN SIMPLE INTEREST (S.I.) :

1. P= Principal or Sum.
2. A = Amount.
3. I= Simple interest.

4. n = No of years for which Principal is used.
 5. r = rate of interest (generally, rate of interest is expressed in percentage)

SOME FORMULAE OF SIMPLE INTEREST :

$$1. I = \frac{P \cdot r \cdot n}{100} \qquad 2. P = \frac{100I}{rn} \qquad 3. r = \frac{100I}{P \cdot n}$$

$$4. n = \frac{100I}{P \cdot r} \qquad 5. A = P + I = P + \frac{P \cdot r \cdot n}{100} = P \left(1 + \frac{rn}{100} \right)$$

Note:

- (i) In the above formulae 'n' indicates no. of years (periods).
 (ii) If time (n) is given in months, convert months in year by dividing the no. of months by 12.
 If time is given in days, convert days by dividing the no. of days by 365.
 If time is given in weeks, divide no. of weeks by 52.

(iii) Rate of interest 5% is $= \frac{5}{100}$

” ” ” $3\frac{1}{2}\%$ is $= \frac{7}{200}$ etc.

We shall now discuss some problems in connection with the above formulae.

Example 1. What sum of money lent out at 6% p.a. will amount of Rs. 620 in 4 years?

Solution : Here, $A = \text{Rs. } 620$, $n = 4$ years, rate of interest $r = 6\%$ p.a.
 $P = ?$

We know, $A = P \left(1 + \frac{rn}{100} \right) = P \left(1 + \frac{6 \times 4}{100} \right)$

$$\Rightarrow 620 = P \cdot \frac{124}{100}$$

$$\therefore P = \frac{620 \times 100}{124} = \text{Rs. } 500$$

Thus, Sum lent out = Rs.500

Example 2. To repay sum of money borrowed 5 months earlier a man agreed to pay Rs. 529.75. Find the sum borrowed if the rate of interest charged was $4\frac{1}{2}\%$ p.a.

Solution : Here, $A = \text{Rs. } 529.75$, $n = 5$ months $= \frac{5}{12}$ year

$$r = 4\frac{1}{2}\% \text{ p.a.}, P = ?$$

$$\begin{aligned} \text{We know, } A &= P\left(1 + \frac{rn}{100}\right) = P\left(1 + \frac{\frac{9}{2} \cdot 5}{100}\right) \\ &\Rightarrow 529.75 = P\left(\frac{2445}{2400}\right) \\ &\Rightarrow \frac{52975}{100} = P\left(\frac{2445}{2400}\right) \\ \therefore P &= \frac{5975 \times 24}{2445} = \text{Rs. } 520 \end{aligned}$$

Thus, Sum borrowed = Rs. 520.

Example 3. What sum will amount to Rs. 5,200 in 6 years at the same rate of S.I. at which Rs. 1706 amount to Rs. 3412 in 20 years?

Solution : Here, A = Rs. 5,200, n = 6 years, (first case)
Let the rate of S.I. be r% p.a.

Second case:

P = Rs. 1706, A = Rs. 3412, n = 20 years

(rate of interest is same as before)

$$\begin{aligned} \therefore \text{S.I.} &= A - P = \text{Rs. } (3412 - 1706) \\ &= \text{Rs. } 1706 \text{ (i.e.I)} \end{aligned}$$

$$\begin{aligned} \text{Now, } I &= \frac{P \cdot r \cdot n}{100} \\ \Rightarrow 1706 &= \frac{1706 \times r \times 20}{100} \quad \therefore r = 5\% \text{ P.a.} \end{aligned}$$

$$\begin{aligned} \text{For Case I} \quad &\Rightarrow A = P\left(1 + \frac{rn}{100}\right) \\ &\Rightarrow \text{Rs. } 5200 = P\left(1 + \frac{5 \times 6}{100}\right) \\ \therefore P &= \frac{5200 \times 100}{130} = \text{Rs. } 4,000 \end{aligned}$$

\therefore Required sum = Rs. 4,000.

Type II : To find rate of interest, Time and S.I.

Example 4. The S.I. on a sum of money at the end of 8 years is $\frac{2}{5}$ th of the sum itself. Find the rate percent p.a.

Solution : Let the sum be Rs. 100, here, n = 8 years

$$\text{S.I.} = \frac{2}{5} \times \text{Rs. } 100 = \text{Rs. } 40$$

Let the rate percent per annum be r .

$$\text{We know, } I = \frac{\text{P.r.n.}}{100}$$

$$\Rightarrow 40 = \frac{100 \times r \times 8}{100}$$

$$\therefore r = 5\%$$

\therefore Required rate of interest = Rs. 5% p.a.

Example 5. A sum of money becomes double in 20 years at S.I. In how many years will it be treble?

Solution : Let the sum be Rs. 100 & rate interest be $r\%$ p.a.

A/q, $n = 20$ years, $A = \text{Rs.}200$

$$\text{We know, } I = P\left(1 + \frac{rn}{100}\right)$$

$$\Rightarrow 200 = 100 \left(1 + \frac{P.20}{100}\right)$$

$$\Rightarrow 200 - 100 = 20r$$

$$\therefore r = 5\%$$

Again, $A = \text{Rs.}300$, $n=?$, $r=5\%$

$$\text{We know, } A = P\left(1 + \frac{rn}{100}\right)$$

$$\Rightarrow 300 = 100\left(1 + \frac{5 \times n}{100}\right)$$

$$\Rightarrow 300 - 100 = 5n$$

$$\therefore n = 40 \text{ years.}$$

Example 6. A man borrowed from his banker Rs.500 for 8 months on Dec.,31. At the end of 5 months, he paid Rs.372 and the remainder with interest on Aug.,31. If his final payment was Rs.137.61; find the rate of interest charged p.a. by the bank.

Solution : A/q, total interest paid = Interest on Rs.500 for 5 months + interest on Rs.(500-372) i.e. on Rs.128 for 3 months.

$$\text{Rs.} \left(\frac{500.r.5}{12 \times 100} + \frac{128.r.3}{12 \times 100} \right), \text{ where, } r\% \text{ p.a. is the rate interest.}$$

$$= \frac{1}{1200} (2500r + 384r)$$

$$= \frac{2884r}{1200}$$

$$\begin{aligned}\text{His total payments in 8 months} &= (372+137.61) \\ &= \text{Rs.}509.61\end{aligned}$$

$$\therefore \text{Total interest paid} = \text{Rs.} (509.61-500) = \text{Rs.} 9.61$$

$$\begin{aligned}\therefore 9.61 &= \frac{2884r}{1200} \quad \therefore r = \frac{961 \times 12}{2884} = \frac{961 \times 3}{721} \\ &= \frac{2883}{721} = 4\%\end{aligned}$$

$$\therefore \text{Rate of interest} = 4\% \text{ p.a. (approx).}$$

Example 7. Find S.I. on Rs. 5600 at 12% p.a. from July 15 to Sept. 26, 2003.

Solution : Here, P = Rs. 5600, r = 12%, n = time = No. of days from July 15 to Sept. 26, 2003

$$= 16+31+26 = 73 \text{ days}$$

$$\begin{aligned}\text{Now, S.I. ie. } I &= \frac{P.r.n}{100} \\ &= \frac{5600 \times 12 \times 73}{365 \times 100} \\ &= \frac{672}{5} \\ &= \text{Rs. } 134.40\end{aligned}$$

$$\therefore \text{Required S.I.} = \text{Rs. } 134.40$$

Example 8. A certain sum of money at S.I. amounts to Rs.560 in 3 years and to Rs.600 in 5 years. Find the sum and the rate of interest.

Solution : Amount after 5 years = Rs. 600
 ” ” 3 ” = Rs. 560

$$\therefore \text{Interest for 2 years} = \text{Rs. } 40$$

$$\therefore \text{Interest for 3 years} = \text{Rs. } 60$$

$$\text{Now, } I = A - P$$

$$\Rightarrow 60 = 560 - P \quad \therefore P = \text{Rs. } 500 = \text{Sum}$$

$$\text{Again } I = \frac{P.r.n}{100}$$

$$= 60 = \frac{500 \times r \times 3}{100}, \text{ whose } r\% \text{ p.a. is the rate of interest.}$$

$$r = \frac{60}{15} = 4\%$$

$$\therefore \text{Required Sum} = \text{Rs. } 500$$

$$\& \text{ Required rate of interest} = 4\% \text{ p.a.}$$

ILLUSTRATIVE EXAMPLES :

Example 9. A man borrowed two sums of money differing by Rs.100 at the same time, one at 5% p.a. and the other at $6\frac{1}{4}\%$ p.a. both at S.I. At the end of 5 years he paid back the loans with interest. If he paid the same amount in respect to each loan, find the sums borrowed by him.

Solution : Let the first sum borrowed by him at 5% p.a. be Rs. X

\therefore the 2nd sum borrowed by him at $6\frac{1}{4}\%$ p.a. = Rs. $(X-100)$

$$\text{Now, Amount for 1st loan} = \text{Rs. } X \left(1 + \frac{5 \times 5}{100}\right) = x \times \frac{5}{4}$$

[\therefore rate of int. in 2nd loan is higher than 1st one & Amt. after the same period for each loan is same.]

$$\text{Amount for 2nd loan} = \text{Rs. } (X-100) \left(1 + \frac{25 \cdot 5}{100}\right) = \text{Rs. } \frac{5X}{4}$$

$$= (X-100) \left(\frac{525}{400}\right) \text{ same.}]$$

$$= \text{Rs. } (X-100) \frac{21}{16}$$

$$\text{A/q, } \frac{21}{16}(X-100) = \frac{5X}{4}$$

$$\Rightarrow \frac{21}{4}(X-100) = 5x$$

$$\Rightarrow 21x - 20x = 2100 \quad \therefore x = \text{Rs. } 2100$$

\therefore Required Sum borrowed at 5% = Rs. 2100

& " " " " $6\frac{1}{4}\%$ = Rs. 2000

Example 10. Arun deposited a total sum of Rs.15,000 in two different banks. One of them pays 3% int. & the other pays $2\frac{1}{2}\%$ int. At the end of the year, he receives Rs.432.75 as interest. Find the sum deposited in each bank.

Solution : Let the deposited sum at 3% int. be Rs.X

\therefore the deposited sum at $2\frac{1}{2}\%$ int. = Rs. $(15000-X)$

$$\text{Total int. received by him after 1 year} = \frac{x \times 3 \times 1}{100} + \frac{(15000-x) \cdot \frac{5}{2} \cdot 1}{100}$$

$$\begin{aligned}
 &= \frac{1}{100} \left\{ 3x + \frac{(15000 - x) \times 5}{2} \right\} \\
 &\Rightarrow \frac{43273}{4} = \frac{1}{100} \times \frac{x + 75000}{2} \\
 &\Rightarrow 43275 \times 2 = \frac{1}{100} \times \frac{x + 75000}{2} \\
 &\quad \therefore x = 86,550 - 75,000 \\
 &\quad \quad = 11,550
 \end{aligned}$$

\therefore Required Sum borrowed at 3% int. = Rs. 11,550

& " " " " " $2\frac{1}{2}$ % int. = Rs. 15000 - 11,550 = Rs. 3,450.

Example 11. Putul borrowed Rs. 75,000 at a certain rate of S.I. for 2 years and Rs. 6,000 at 1% higher rate of S.I. than the first for 1 year. He paid Rs. 2580 for both loans as interest in all. Find the two rates of interest.

Solution : Let the original rate of int. be $r\%$ p.a.

$$\text{S.I. on Rs. 7500 for 2 years @ } r\% \text{ p.a.} = \text{Rs. } \frac{75000 \times r \times 2}{100}$$

A/q, rate of int. on 2nd loan = $(r+1)\%$

$$\begin{aligned}
 \therefore \text{ S.I. on Rs. 6000 for 1 year} &= \text{Rs. } \frac{6000 \times (r+1) \times 1}{100} \\
 &= \text{Rs. } 60 (r+1)
 \end{aligned}$$

$$\text{A/q, } 150r + 60 (r+1) = 2580$$

$$\therefore r = \frac{2520}{210} = 12\%$$

\therefore Required rates of int. are 12% & 13%.

Example 12. If S.I. on Rs. 1800 exceeds the interest on Rs. 1650 in 3 years by Rs. 45, at the same rate of interest, find the rate of interest p.a.

Solution : Let the original rate of int. be $r\%$ p.a.

Difference of int. on Rs. 1800 and Rs. 1650 at the rate $r\%$ p.a. for

$$3 \text{ years} = \left(\frac{1800 \times r \times 3}{100} - \frac{1650 \times r \times 3}{100} \right) = \text{Rs. 45 (given)}$$

$$\Rightarrow \frac{3r}{100} \times 150 = 45 \quad \therefore r = \frac{450}{45} = 10\%$$

\therefore Required rate int. p.a = 10%.

Example 13. The difference in S.I. on a certain sum of money at 14.5% p.a. in 3 years and $4\frac{1}{2}$ years is Rs.696. Find the sum.

Solution : Let the sum be Rs.100

Difference in int. on the sum Rs.100 for $4\frac{1}{2}$ years and 3 years

$$\begin{aligned} @ 14.5\% \text{ p.a.} &= \text{Rs.} \left(\frac{100 \times 29 \times 9}{2 \times 100 \times 2} - \frac{100 \times 29 \times 3}{2 \times 100} \right) \\ &= \frac{261}{4} - \frac{87}{2} = \text{Rs.} \frac{87}{4} \end{aligned}$$

Differance (Rs.) Sum (Rs.)

$$\frac{87}{2} \quad 100$$

$$\begin{aligned} 696 \quad x \quad \therefore x &= 100 \times \frac{2}{87} \times 696 \\ &= \text{Rs.} 1600 \end{aligned}$$

\therefore Required sum = Rs. 1600.

Example 14. A person borrowed Rs.6400. After 2 years and 3 months he paid Rs.6136 in cash plus a bicycle and hence cleared up the entire amount borrowed. If the rate of S.I. was $3\frac{1}{2}\%$ p.a., find the value of the bicycle.

$$\begin{aligned} \text{Solution :} \quad A &= P \left(1 + \frac{rn}{100} \right) = \text{Rs.} 6400 \left(1 + \frac{7 \times 2\frac{1}{4}}{100} \right) \\ &= \text{Rs.} 6400 \left(1 + \frac{63}{800} \right) \\ &= \text{Rs.} 8 \times 863 \\ &= \text{Rs.} 6904 \end{aligned}$$

A/q, value of bicycle = Rs. (6904–6136) = Rs.768.

Example 15. A man left Rs.1,30,000 for his two sons aged 10 years and 16 years with the direction that the sum should be divided in such a way that the two sons got the same amount when they attain 18 years. How much did the elder son get in the beginning, rate of int. is $12\frac{1}{2}\%$ p.a.?

Solution : His elder son will attain the age of 18 years after 2 years
& his younger son will attain the age of 18 years after 8 years

Let the share of elder son at the beginning be Rs. x

\therefore Let the share of younger son at the beginning be Rs. (1,30,000–x)

$$\begin{aligned}\text{Amount due to elder son after 2 years} &= x\left(1 + \frac{25 \times 2}{2 \times 100}\right) \\ &= \text{Rs. } \frac{5x}{4}\end{aligned}$$

$$\begin{aligned}\text{Amount due to younger son after 8 years} &= (1,30,000 - x)\left(1 + \frac{25 \times 8}{2 \times 100}\right) \\ &= (1,30,000 - x) \times 2\end{aligned}$$

$$\text{A/q, } \frac{5x}{4} = 2(1,30,000 - x)$$

$$\therefore x = \frac{8 \times 1,30,000}{13} = \text{Rs. } 80,000$$

Hence, Share of elder son at the beginning = Rs.80,000.

EXERCISE

1. Sunil borrowed Rs.7500 at 14.5% p.a. for $2\frac{1}{2}$ years. Find the amount he had to pay after that period.
(Ans: Rs.10218.75)
2. A sum of Rs.1200 was lent out for 2 years at S.I. The lender got Rs.1536 in all. Find the rate int. p.a.
(Ans: Rs.14%)
3. In what time Rs.5000 will yield Rs.1100 @ $5\frac{1}{2}\%$?
(Ans: 4 years)
4. In a certain time Rs.1200 becomes Rs.1560 at 10% p.a. S.I. Find the Principal that will become Rs.2232 at 8% p.a. in the same time.
(Ans: Rs.1800)
5. A person invested some amount of money at 12% p.a. S.I. for $2\frac{1}{2}$ years and some amount at 12.5% p.a. S.I. for 2 years. If he had an amount of Rs. 10,000 in hand and on such investment earned Rs.2700 in all, find the amount invested in each case.
(Ans: Rs. 4000, Rs. 6,000)
6. A man deposited Rs.5000 on 20th April in a company, paying int. at 2% pa S.I. He withdrawns Rs.3000 on 15th May and deposited Rs.4000 on 6th June. How much interest was due to him on 30th June following?
(Ans: Rs.17.21 approx)

7. A sum of money amounts to Rs. 4720 in 2 years and Rs.5260 in $3\frac{1}{2}$ years at S.I. Find the sum and rate of int.
(Ans: Rs.4000, 9%)
8. A person borrowed at the same time Rs.250 and Rs.350 at 6% and 5% pa S.I. respectively on condition that the entire loan and accrued interest be paid when the amount of both shall be Rs.730. How long is the loan to continue?
(Ans: 4 years)
9. The rates of S.I. in two banks A and B are in the ratio 5:4. A person wants to deposit his total savings in the two banks in such a way that he received equal half yearly S.I. from both. In what ratio he should deposit his savings in banks A & B?
(Ans: 4:5)
10. If Rs.1680 amounts to Rs.2352 in $7\frac{1}{2}$ years. In how many years will Rs.1350 amount to Rs.1782, the rate of S.I. being same in both the cases.
(Ans: 6 years)
11. At what rate of S.I. pa will Rs.1500 produce the same interest in 5 years as Rs.3125 produce in 3 years at 4% pa.
(Ans: Rs.5%)
12. Amit and Bipin borrowed Rs.800 each at the same rate of S.I. for 3 years and 6 years respectively. If Amit pays back Rs.1052 to clear his debt, find the amount which Bipin has to pay.
(Ans: Rs.1304)

COMPOUND INTEREST

DIFFERENCE BETWEEN SIMPLE AND COMPOUND INTEREST :

For every year, S.I. is always calculated on the original principal upto the total period for which the sum (principal) is used.

But in case of C.I., the interest as soon as it is due after a certain period, is added to the principal and interest for the succeeding period is based upon the principal and interest added together. Hence the principal does not remain fixed at the original sum but increases at the end of each interest period.

Interest period is the period after which, the interest becomes due. A year is generally taken as interest period, but for practical reasons it may be half-year, a quarter year or month.

CALCULATION OF C.I. WITHOUT USING LOGARITHON : (Ordinary method)

Example : Find C.I. on Rs.500 for 3 years at 5% pa, interest being compound annually.

Solution :	Original Principal	= Rs.500	
	Int. for 1st year	= Rs. 25	(5% of Rs.500 = Rs.25)
	<hr/>		
	Principal for 2nd yr.	= Rs.525	
	Int. for 2nd year	= Rs.26.25	(5% of Rs.525 = Rs.26.25)
	<hr/>		
	Principal for 3rd yr.	= Rs.551.25	
	Int. for 2nd year	= Rs.27.5625	(5% of Rs.551.25 = Rs.27.5625)
	<hr/>		
	Amount at the end of 3rd yr.	= Rs.578.8125	
	Now, C.I. = A-P = Rs.(578.8125-500) = Rs.78.8125		
		= Rs.78.81	

Note:

- (i) Calculation should be made upto 4 decimal places.
- (ii) When no. of yrs. is large and rate of int. is in fraction then it becomes difficult to do a problem of C.I. by ordinary method as discussed above. On the other hand, in such cases, use of logarithms facilitates easy calculation.

PROOF OF COMPOUND INTEREST FORMULA :

Let the principal be Rs. P, no. of years be 'n', rate of int. be r% pa, (int. being compounded annually) & the amount after 'n' years be Rs. A.

Let i = Int. on Re.1 for 1 year (ie. $i = \frac{r}{100}$)

Original Principal = Rs.P

Int. for 1st year on Rs.P = Rs.P.i.

Principal at the beginning of 2nd yr. = $P(1+i)$ = Amt. at the end of 1st yr.

Int. for 2nd year = $P(1+i).i$

Principal at the beginning of 3rd yr. = $P(1+i)+P(1+i).i$

= $P(1+i)(1+i)$

= $P(1+i)^2$ = Amt. at the end of 2nd yr.

& soon

∴ Amount at the end of 'n' years will be:

$$A = P(1+i)^n \dots\dots\dots(1)$$

When int. is payable half yearly, formula (1) reduces to :

$$A = P\left(1 + \frac{i}{2}\right)^{2n} \dots\dots\dots(2)$$

When int. is payable quartaly, (2) reduces to :

$$A = P\left(1 + \frac{i}{4}\right)^{4n} \dots\dots\dots(3)$$

When int. is payable monthly (3) reduces to :

$$A = P\left(1 + \frac{i}{12}\right)^{12n} \dots\dots\dots(4)$$

EFFECTIVE RATE OF INTEREST :

If the rate of int. is 5% pa, payable yearly then int. o Rs.100 for 1 year is Rs.5. So that amount becomes Rs.105 after 1 year.

Again if the int. is payable half-yearly, then the principal becomes Rs.102.50 after first half-year (6 months). For the 2nd half-year int. will be counted by on Rs.102.50 & hence total amount will be Rs. (102.50+2.56) = Rs.105.06.

$$\begin{aligned} \therefore \text{Effective rate (percent per year)} &= \text{Rs. } (105.06-100) \\ &= \text{Rs. } 5.06 \end{aligned}$$

Note:

5% is known as flat rate or nominal rate where as the true rate i.e. 5.06% is the effective rate.

FORMULA FOR EFFECTIVE RATE OF INTEREST :

$$\text{Effective rate} = 100 \left\{ \left(1 + \frac{i}{p} \right)^p - 1 \right\}$$

Where, i = rate of int. per unit per period
 p = No. of int. periods

DEPRECIATION :

Depreciation is the loss in the value of an asset wear and tear with the passage of time. Loss in the value of an asset occurs due to constant use or sun rays or due to any of the natural causes.

In case of depreciation, the principal value is diminished every year a certain amount & in the subsequent period, the diminished value becomes the principal value.

Thus we find that in case of compound interest, int. is in creasing in every period where as in case of depreciation value of the asset is degreasing every period at a constant rate.

Hence, the depreciation formula takes the following shape

$$A = P(1-i)^n \text{ where, } \begin{array}{l} i = \text{Depreciation Re 1 for 1 year} \\ P = \text{Original value of an asset} \\ A = \text{Scrap value} \\ n = \text{No. of yrs. (Periods)} \end{array}$$

We shall now discuss some problems in connection with the above formula.

TYPE I: TO FIND C.I.

Example 1. Find the C.I., the int. compounded annually on Rs.2500 for 4 years at $3\frac{1}{2}\%$ pa.

Solution : Here, $P = \text{Rs.}2500$, $n = 4$ years, $r = 3\frac{1}{2}\%$ $\therefore i = \frac{3\frac{1}{2}}{100} = 0.035$ C.I. =?

$$\begin{aligned} \text{We know, } & A = P(1+i)^n \\ \Rightarrow & A = 2500 (1.035)^4 \end{aligned}$$

Taking log on both sides we get

$$\begin{aligned} \log A &= \log 2500 + 4 \log 1.035 \\ &= 3.3979 + 4 \times 0.0149 \\ &= 3.3979 + .0596 \\ &= 3.4575 \end{aligned}$$

$$\therefore A = \text{anti log } 3.4575 = \text{Rs. } 2867$$

$$\text{Now, CI} = A - P = \text{Rs.}(2867 - 2500) = \text{Rs. } 367$$

Note:

If nothing is mentioned about the modality of payment of interest, consider interest compounded annually.

Example 2. Find the C.I. on Rs.800 for 2 years, if interest be payable half-yearly the rate for first year being 5%pa and for the second year 6% pa.

Solution : Here, amount after 1st year = $800(1.025)^{2 \times 1} = 800 \times 1.050625$
 $= 840.5000$
 $= 840.50$

\therefore C.I. for 1st year = Rs.(840.50–800) = Rs.40.50

Amount after 2nd year = $840.5(1.03)^{2 \times 1} = 840.5 \times 1.0609$
 $= 891.69$

\therefore C.I. for 2nd year = Rs.(891.69–840.50)

= Rs.51.19 Here, Formula used $A = P \left(1 + \frac{i}{2}\right)^{2n}$

\therefore Total C.I. for 2nd year = Rs.(40.50+51.19)

= Rs.91.69

TYPE II: TO FIND AMOUNT

Example 3. Find the amount due on Rs.1000 for 4 years at 5% pa. C.I.

Solution : Here, P = Rs.1000, n = 4 years, $i = \frac{5}{100} = 0.05$ A=? C.I.=?

We know, $A = P(1+i)^4$

$$\Rightarrow A = 1000 (1.05)^4$$

Taking on both sides we get

$$\begin{aligned} \log A &= \log 1000 + 4 \log 1.05 \\ &= 3 + 4 \times 0.0212 \\ &= 3.0848 \end{aligned}$$

$\therefore A = \text{anti log } 3.0848 = 1215$

\therefore Required amount = Rs.1215 & C.I. = A–P
 $= \text{Rs.}(1215–1000)$
 $= \text{Rs.}215$

TYPE III: TO FIND PRESENT VALUE OR SUM OR PRINCIPAL

Example 4. What sum of money put out at 4% pa C.I. for 18 years will amount to Rs.10,000?

Solution : Here, $A = \text{Rs.}10,000$, $n = 18$, $i = \frac{4}{100} = 0.04$ $P = ?$

$$\begin{aligned} \text{We know, } \quad A &= P(1+i)^n \\ \Rightarrow 10000 &= P(1.04)^{18} \end{aligned}$$

Taking on both sides we get

$$\begin{aligned} \log 10,000 &= \log P + 18 \log 1.04 \\ \Rightarrow \log P &= 4 - 0.3060 \\ &= 3.6940 \end{aligned}$$

$$\therefore P = \text{anti log } 3.694 = 4943$$

$$\therefore \text{Required amount} = \text{Rs.}4943$$

Example 5. What sum will amount to Rs.1401.60 in 2 years at 5% pa C.I., payable half-yearly?

Solution : We know, $A = P\left(1 + \frac{i}{2}\right)^{2n}$

$$\Rightarrow 1401.60 = P\left(1 + \frac{0.05}{2}\right)^{2 \times 2} = P(1.025)^4$$

Taking on both sides we get

$$\begin{aligned} \log 1401.6 &= \log P + 4 \log 1.025 \\ \Rightarrow \log 1402 &= \log P + 4 \times 0.0107 \\ \Rightarrow 3.1467 &= \log P + .0428 \end{aligned}$$

$$\therefore P = \text{Anti log } 3.1039 = 1271$$

$$\therefore \text{Required amount} = \text{Rs.}1271$$

TYPE IV: TO FIND RATE OF INTEREST

Example 6. At what rate of interest will Rs.6345 amount to Rs.7288 in 7 years, int. being compounded annually?

Solution : Here, $A = \text{Rs.}7288$, $P = \text{Rs.}6345$, $n = 7$, $r = ?$

$$\begin{aligned} \text{We know, } \quad A &= P(1+i)^n, \text{ int. being compounded annually} \\ \Rightarrow 7288 &= 6345(1+i)^7 \end{aligned}$$

Taking on both sides we get

$$\begin{aligned} \log 7288 &= \log 6345 + 7 \log (1+i) \\ \Rightarrow 3.8626 &= 3.8024 + 7 \log (1+i) \end{aligned}$$

$$\Rightarrow \log(1+i) = \frac{0.0602}{7} = 0.0086$$

$$\therefore 1+i = \text{Anti log } 0.0086 = 1.020$$

$$\therefore 1+i = 1.020 - 1 = 0.020$$

$$\begin{aligned} \text{Now, } i &= \therefore r = 100i = 100 \times 0.020 \\ &= 2\% \end{aligned}$$

\therefore Required rate of int. = 2%.

Example 7. Find the rate percent at which a certain sum of money invested at C.I. doubles itself in 17 years?

Solution : Let the sum = Rs.100 = P

A/q, A = Rs.200, here n=17, r=?

We know, $A = P(1+i)^n$

$$\Rightarrow 200 = 100(1+i)^{17}$$

$$\Rightarrow 2 = (1+i)^{17}$$

Taking on both sides we get

$$\log^2 = 17 \log(1+i)$$

$$\Rightarrow 0.3010 = 17 \log(1+i)$$

$$\therefore \log(1+i) = \frac{0.3010}{17} = 0.0177$$

$$\therefore i = 0.042$$

$$\begin{aligned} \text{Now, } r &= 100i = 100 \times 0.042 \\ &= 4.2\% \end{aligned}$$

\therefore Required rate of int. = 4.2%.

Example 8. What is the rate percent pa if Rs.6000 amounts to Rs.10,000 in 15 years, int. being compounded half-yearly.

Solution : Here, P=Rs.6000, A=Rs.10,000, n=15, r=? (Int. compounded half-yearly)

We know, $A = P\left(1 + \frac{i}{2}\right)^{2n}$

$$\Rightarrow 10,000 = 6000\left(1 + \frac{i}{2}\right)^{2 \times 15}$$

$$\Rightarrow 10 = 6\left(1 + \frac{i}{2}\right)^{30}$$

Taking on both sides we get

$$\begin{aligned} \log 10 &= \log^6 + 30 \log \left(1 + \frac{i}{2}\right) \\ \Rightarrow 10 &= 0.7782 + 30 \log \left(1 + \frac{i}{2}\right) \\ \Rightarrow \frac{0.2218}{30} &= \log \left(1 + \frac{i}{2}\right) \\ \Rightarrow 0.0074 &= \log \left(1 + \frac{i}{2}\right) \\ \therefore 1 + \frac{i}{2} &= \text{Anti log } 0.0074 = 1.017 \\ \Rightarrow i &= 0.017 \times 2 = 0.034 \\ \therefore \text{Required rate of int.} &= 3.4\%. \end{aligned}$$

TYPE V: TO FIND TIME

Example 9. In what time will a sum of money double itself at 5% pa C.I. payable half-yearly?

Solution : Let the sum be Rs. 100 = p (say), Here, $i = \frac{5}{100} = 0.05$ (int. compounded half yearly) A/q, amount = Rs. 200 = A (say)

$$\begin{aligned} \text{We know, } A &= \left(1 + \frac{i}{2}\right)^{2n} \\ \Rightarrow 200 &= 100 = \left(1 + \frac{.05}{2}\right)^{2n} = 100(1.025)^{2n} \\ \Rightarrow 2 &= (1.025)^{2n} \end{aligned}$$

Taking log on both sides we get

$$\begin{aligned} \log 2 &= 2n \log 1.025 \\ \Rightarrow 0.3010 &= 2n \times 0.0107 \end{aligned}$$

$$\therefore n = \frac{0.3010}{.0214} = \frac{3010}{214} \cong 14.2 \text{ yr.}$$

ILLUSTRATIVE EXAMPLES :

Example 10. The difference between C.I. & S.I. on a sum of money lent for 2 years at 5% pa is Rs. 27.50. Find the sum that was lent.

Solution : Let the sum be Rs. 100.

S.I. for 2 years @5% pa = Rs.10, Here, $i = \frac{5}{100} = 0.05$, $n = 2$ yrs.

Let us find amount for 2 yrs. @5% pa

$$\begin{aligned}\text{We know, } A &= (1+i)^2 \\ &= 100 (1.05)^2 \\ &= 100 (1.1025) = \text{Rs. } 110.05\end{aligned}$$

Now, CI = A - P = Rs. (110.25 - 100) = Rs. 10.25

\therefore Difference let the CI & S.I. = Rs. (10.25 - 10) = Rs. 0.25

Difference (Rs.)	Sum (Rs.)
0.25	100
27.50	x

$$\begin{aligned}\therefore x &= 100 \times \frac{27.50}{0.25} = \frac{100 \times 2750}{25} \\ &= 11,000\end{aligned}$$

\therefore Required Sum = Rs. 11,000.

Example 11. The S.I. and C.I. on a certain sum of money for 2 years at a certain rate are Rs. 90 and Rs. 93 respectively. Find the sum and the rate.

Solution: S.I. for 1 year = Rs. 45, Difference of C.I. & S.I. = Rs. 3
 \therefore Interest on Rs. 45 for 1 year = Rs. 3

$$\therefore \text{Interest on Rs. } 100 \text{ for 1 year} = \frac{3}{45} \times 100 = \text{Rs. } 6\frac{2}{3}$$

$$\therefore \text{Rate of Interest} = 6\frac{2}{3} \%$$

$$\begin{aligned}\text{Again, } I &= \frac{\text{P} \cdot \text{r} \cdot \text{n}}{100} \\ = 90 &= \frac{\text{P} \cdot \frac{20}{3} \times 2}{100} \therefore \text{P} = \text{Rs. } 675\end{aligned}$$

\therefore Required Sum = Rs. 675 & Rate of Int. = $6\frac{2}{3} \%$ pa.

Example 12. A sum of money double itself in $12\frac{1}{2}$ years at a certain rate of S.I. How long will it take to double itself at the same rate of C.I.?

Solution: Let the sum be Rs. 100 = P

$$\text{A/q, } A = \text{Rs. } 200, n = \frac{25}{2} \text{ year, Let the rate be } r\% \text{ pa. } i = \frac{r}{100}$$

$$\therefore \text{S.I.} = I = \text{Rs. } (200 - 100) = \text{Rs. } 100$$

$$\text{Now, } I = \frac{p.r.n.}{100}$$

$$\Rightarrow 100 = \frac{100 \times r \times \frac{25}{2}}{100} \quad \therefore r = 8\%$$

Again, $P = \text{Rs.}100$, $A = \text{Rs.}200$, $r = 8\%$ pa C.I., $n = ?$

$$\therefore i = \frac{8}{100} = 0.08$$

$$\text{Now, } A = P(1+i)^n$$

$$\Rightarrow 200 = 100 (1.08)^n$$

$$\Rightarrow 2 = (1.08)^n$$

Taking log on both side we get

$$\log 2 = n \log 1.08$$

$$\Rightarrow 0.3010 = n \times 0.0334 \quad \therefore n = \frac{3010}{334} \\ \simeq 9 \text{ years}$$

\therefore Required Time = 9 years (approx).

Example 13. C.I. on a certain sum of money for 3 years @10% pa. is Rs.993. What would be S.I. at the same rate on the same sum for the same time?

Solution: A/q, $P(1+0.10)^3 - P = 993$

$$\Rightarrow P\{(1.1)^3 - 1\} = 993$$

$$\Rightarrow P\{(1.1)^3 - 1\}$$

$$\Rightarrow P\{0.331\} = 993 \quad \therefore P = \frac{993000}{331} = \text{Rs.}3000$$

\therefore Sum = Rs.3000

$$\text{Now, } I = \frac{p.r.n.}{100} = \text{Rs.} \frac{3000 \times 10 \times 3}{100} = \text{Rs.}900.$$

\therefore Required S.I. = Rs.900.

Example 14. C.I. for 2nd year on a certain sum at 4% pa is Rs.25. Find C.I. for 3rd year.

Solution:

Let the sum be	=	Rs.100
Int. for 1st year	=	Rs. 4
Principal for 2nd year	=	Rs. 104
Int. for 2nd Year	=	Rs. 4.16 (4% of Rs. 104)
Principal for 3rd year	=	Rs.108.16
Int. for 3rd year	=	Rs.4.3264 (4% of Rs.108.16)

Int. for 2nd yr. (Rs.) Int. for 3rd yr. (Rs.)

$$\begin{array}{ccc} 4.16 & & 4.3264 \\ 25 & & x \end{array}$$

$$\begin{aligned} \therefore x &= 4.3264 \times \frac{25}{4.16} = \frac{43264 \times 25}{41600} \\ &= \frac{43264}{1664} \\ &= \text{Rs. } 26 \end{aligned}$$

Hence, Required 3rd yr. int. = Rs.26.

Example 15. A sum of money amount to Rs.8820 after 2 years and Rs.9261 after 3 years at a certain rate of C.I. Find the sum and rates of interest.

Solution: Amount after 3 years = Rs. 9261

Amount after 2 years = Rs. 8820

$$\therefore \text{Difference} = \text{Rs. } 441$$

$$\therefore \text{Interest on Rs. 8820 for 1 year} = \text{Rs. } 441 \text{ (Note this step)}$$

$$\therefore \text{Interest on Rs. } 100 \text{ for 1 year} = \frac{441}{8820} \times 100 = \text{Rs. } 5$$

$$\therefore \text{Rate of int.} = 5\% \text{ pa}$$

$$\text{Again, } A = P(1+i)^n \quad i = \frac{5}{100} = 0.05$$

$$\begin{aligned} 8820 &= P(1.05)^2 \quad \therefore P = \frac{8820}{1.1025} \\ &= \frac{8820.0000}{11025} \\ &= \text{Rs. } 8000 \end{aligned}$$

$$\therefore \text{Sum} = \text{Rs. } 8000 \text{ \& rate of int.} = 5\%.$$

Example 16. Borrowing a certain sum of money at 4% S.I., a person invested the entire amount at 5% C.I. for 3 years and gained Rs. 376.25. Find the sum borrowed.

Solution: Let the sum borrowed by = Rs.100
 S.I. on Rs.100 for 3 years = Rs.4×3 = Rs.12
 C.I. for 3 years @5% on Rs.100 = $100(1.05)^3 - 100$
 = Rs.15.76

$$\therefore \text{Gain} = \text{Rs.}(15.76 - 12) = \text{Rs. } 3.76$$

Gain	Sum borrowed (Rs.)
3.76	100
376.25	x

$$\begin{aligned} \therefore x &= 100 \times \frac{376.25}{3.76} \\ &= \frac{100 \times 376.25}{3.76} = \text{Rs.}10000 \end{aligned}$$

\therefore Sum borrowed = Rs.10,000 approx.

EXERCISE

- What sum will amount to Rs.1000 in 12 years at $4\frac{1}{2}\%$ pa CI?
(Ans: Rs.589.90)
- What sum will amount to Rs.2149 in 5 years at 12% pa C.I. payable half yearly.
(Ans: Rs.1200)
- The difference between S.I. & C.I. on a certain sum put out for 5 years at 3% was Rs.46.80. Find the sum
(Ans: Rs.5200)
- If the population of a town increases every year by 2% of the population at the beginning of that year, in how many years will the total increase of population be 40%?
(Ans: 17 years)
- Prabin borrowed Rs.6000 from a money lender but he could not pay any amount in a period of 4 years. Accordingly, the money lender demanded now Rs.7500 from him. What rate percent pa C.I. did the lender require for lending his money?
(Ans: 5.7%)
- C.I. on a sum at 5% for the 3 years is Rs.158. Find C.I. on the same sum at 6% for 2 years.
(Ans: Rs.124)
- C.I. & S.I. on a certain sum of money for 2 years are respectively Rs.920.25 and Rs.900. Find the sum and rate of interest.
(Ans: Rs.10000, $4\frac{1}{2}\%$)

8. A sum of money invested at C.I. payable yearly amounts to Rs.10816 at the end of 2nd year and to Rs.11248.64 at the end of third year. Find the sum and rate of interest.
(Ans: Rs.10,000 & 4%)
9. A man left Rs.18,000 with the direction that it should be divided in such a way that his 3 sons aged, 9,12 & 15 years should receive the same amount when they reach the age of 25 years. If the rate of C.I. be $3\frac{1}{2}\%$ pa, what should each son receive when he is 25 years old?
(Ans: Rs.9,341)
10. A machine depreciates at the rate of 10% of its value of the beginning of a year. The machine was bought for Rs.5810 & the scrap value realised, when sold, was Rs.2250. Find the number of years the machine was used.
(Ans: 9 years approx)
11. A machine depreciates 10% pa for 1st two years and then 7% for the next three years, depreciation being calculated on the diminishing value. If the value of the machine be Rs.10,000 initially, find the effective rate of depreciation & the depreciated value of the machine at the end of 5th year.
(Ans: 6.968% & Rs.3484)
12. Find the difference but S.I. & C.I. on Rs.1000 for 2 years at 4% payable quarterly.
(Ans: Rs.2)
13. In how many years Rs.3495 will amount to Rs.4680 at 6% pa C.I.?
(Ans: 5 years approx)

Important Note :

When rates of interest are different for different periods

$$(1) \text{ Amount after 1 year} = P\left(1 + \frac{r_1}{100}\right), r_1 \rightarrow \text{rate of int. for 1st year}$$

$$(2) \text{ Amount after 2 year} = P\left(1 + \frac{r_1}{100}\right)\left(1 + \frac{r_2}{100}\right), r_2 \rightarrow \text{rate of int. for 2nd year}$$

$$(3) \text{ Amount after 3 year} = P\left(1 + \frac{r_1}{100}\right)\left(1 + \frac{r_2}{100}\right)\left(1 + \frac{r_3}{100}\right), r_3 \rightarrow \text{rate of int. for 3rd year}$$

ANNUITY

INTRODUCTION :

Items like car, land and building, Computer, refrigerator, motor-bike etc are costly. It is not possible to buy them by an ordinary customer paying the total cost at one time. For this purpose, the traders offer various schemes to customers to promote sales of such costly items. Instead of paying the cost of items in full at the initial stage the customers are given the opportunity to pay the price on instalment basis and as such the customers are attracted to such schemes. The buyer and the seller are both benefitted. The buyer gets the item by paying small amount of money in instalments and enjoy the benefit. In addition, the buyer utilises the assets in earning revenue & he can pay the instalments out of such earnings. Thus, the buyer can pay the instalment while he earns. On the other hand, these schemes help promotion of sales of such costly items. This sales model makes the sale of costly items affordable to the ordinary buyers.

The services of small amount of money paid periodically by the buyer to the seller is called Annuity payments.

DEFINITION OF ANNUITY :

An annuity is a series of regular annual payments of a fixed sum. Instead of paying a lump-sum annually, the interval of payment may be subdivided into 6 months, 3 months or even one month. This subdivision of interval may be helpful to the annuitant for easy payments. The person who receives the annuity is called Annuitant.

The total time for which an annuity is to continue is called its duration. Each equal payment is called instalment. In case of annuity, C.I. is charged.

TYPES OF ANNUITIES :

Broadly speaking, annuities are of three types viz (a) Annuity certain, (b) Perpetual annuity & (c) Annuity Contingent.

(a) In case of Annuity certain, instalments are paid upto a finite and definite number of years. Annuity certain are of three types :

- (1) Annuity immediate
- (2) Annuity due
- (3) Deferred annuity.

In case of Annuity immediate, instalments are paid at the end of each year. Where as, in case of Annuity due, instalments are paid at the beginning of each year.

(a) Deferred annuity : It is such an annuity which does not begin until the lapse of a certain period & after it starts, it continues regularly upto a certain period. eg. if an annuity is deferred for m years, then 1st instalment is given after $(m+1)$ years & may continue upto 'n' years. But if payment continues for ever, the annuity is called deferred perpetuity.

(b) Perpetual annuity or Perpetuity : If an annuity is to continue for ever, it is known as perpetual annuity eg. corporation tax, land taxes etc. are perpetual annuities.

(c) Contingent annuity : In this case, annual payments are to continue till the happening of some contingent event such as death of annuitant, marriage of daughter etc.

WHAT IS SINKING FUND?

It is a fund created by depositing regularly a certain sum of money upto a certain period in order to discharge a future liability with accumulated fund. Such a fund is known as **Sinking Fund**.

SOME IMPORTANT SYMBOLS USED IN ANNUITY :

A = Value of each instalment.

n = No. of years (Periods)

i = Interest on Re 1 for 1 year, where rate of int. is $r\%$ pa.

$$\text{ie. } i = \frac{r}{100}$$

v = Present value of an annuity.

= Sum of present values of all instalments.

M = Amount of an annuity

= Sum of amounts of all instalments.

Some formulae on Present values and Amount deduced from sum of a GP.

What is GP?

Geometric progression or GP is a series of quantities, called terms, when the ratio of any term (except the first) to the preceding one is constant. This ratio is known as common ratio (CR)

eg (1) Series is : 1,3,9,27, here 1st term = 1 & CR = 3

(2) Series is : $\frac{1}{2}, 1, 2, 4, \dots$ here 1st term = $\frac{1}{2}$, & CR = 2

(3) Series is : $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \dots$ here 1st term = $\frac{2}{3}$, & CR = $\frac{1}{2}$

General form of a GP is a, ar, ar^2, \dots , here 1st term = a, CR = r, n^{th} term = $ar^{n-1} = t_n$

Sum to n terms of general GP is :

$$S_n = \frac{a(1-r^n)}{1-r}, \text{ when } r < 1$$

$$= \frac{a(r^n - 1)}{r-1}, \text{ when } r > 1$$

AMOUNT OF AN ANNUITY (left unpaid for n years)

Let value of each yearly instalment be = Rs. A

i = Int. on Re 1 for, one year ,r% being rate of C.I.

Let the annuity be left inpaid for n years.

Now, 1st instalment payable after one year earns int. for (n-1) years, as it is due at the end of 1st year. Similarly the second instalment earns int. for (n-2) yrs. & so on. The last instalment earns no int.

∴ Amount of 1st instalment	=	A (1+i) ⁿ⁻¹
” ” 2nd ”	=	A(1+i) ⁿ⁻²
.....		
” ” last instalment	=	A

Now, M = Sum of above amounts = Amount of the annuity
 = A(1+i)ⁿ⁻¹ + A(1+i)ⁿ⁻² + + A(1+i)² + A(1+i) + A
 = A[1+(1+i)+(1+i)² + + A(1+i)ⁿ⁻¹], writing in reverse order.

The series within third bracket is a GP with

1st term = 1 & CR = (1+i) > 1

$$\therefore M = A \cdot \frac{1 \cdot \{(1+i)^n - 1\}}{1+i-1}$$

This, M = $\frac{A}{i} \{(1+i)^n - 1\}$ (1), when int. is payable yearly.

Note:

- (1) is Amount formula for immediate annuity & it is also the formula for sinking fund.

AMOUNT OF AN ANNUITY DUE :

The formula is : M = (1+i) · $\frac{A}{i} \{(1+i)^n - 1\}$ (2) (Here, instalment is payable yearly)

Note:

Here payments are made at the beginning of the period so that each instalment earns int. for a period more. i.e. The 1st instalment earns int. for n years, 2nd instalment for (n-1) yrs. & so on.)

PRESENT VALUE OF IMMEDIATE ANNUITY FOR N YEARS

Let the value of each yearly instalment be Rs. A & int. on Re 1 for 1st yr = i.

Then the present values of 1st, 2nd, 3rd, nth instalments are :

$$\frac{A}{1+i}, \frac{A}{(1+i)^2}, \frac{A}{(1+i)^3}, \dots, \frac{A}{(1+i)^n}$$

$$\text{Here, } A = P_1(1+i)$$

$$\therefore P_1 = \frac{A}{1+i}$$

$$A = P_2(1+i)^2$$

$$\therefore P_2 = \frac{A}{(1+i)^2}$$

P_1, P_2 etc. etc. are present values of 1st, 2nd,..... instalments.

$\therefore V =$ Sum of above present values = Present value of the annuity.

$$= \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \dots + \frac{A}{(1+i)^n}$$

$$= \frac{A}{(1+i)} \left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{n-1}} \right]$$

The series within third bracket is a GP with 1st term = 1 & CR = $\frac{1}{(1+i)} < 1$.

$$= \frac{A}{(1+i)} \left[\frac{1 \cdot \left\{ 1 - \frac{1}{(1+i)^n} \right\}}{1 - \frac{1}{(1+i)}} \right]$$

$$\therefore V = \frac{A}{i} \left[1 - \frac{1}{(1+i)^n} \right], \text{ after simplification(3)}$$

Another form : $V = \frac{A}{i} [1 - (1+i)^{-n}] \dots\dots\dots(4)$

Formula for Present value of Annuity due will be :

$$V = \frac{A}{i}(1+i) [1 - (1+i)^{-n}] \text{ from formula (3)}$$

PRESENT VALUE OF PERPETUITY :

The formula is : $V = \frac{A}{i} \{1 - 0\} = \frac{A}{i}$

$$\therefore V = \frac{A}{i} \dots\dots\dots(5)$$

PRESENT VALUE OF DEFERRED ANNUITY :

(deferred for m years & to continue for n years)

Since the annuity begins at the end of m years, the first instalment is due at the end of (m+1) years, second instalment is due at the end of (m+2) years & so on

Where,

A = Value of each instalment

i = Int. on Re. 1 for 1 year

$$\therefore \text{Present value of A at the end of (m+1) years} = \frac{A}{(1+i)^{m+1}}$$

$$\text{'' '' '' '' '' '' '' '' (m+2) ''} = \frac{A}{(1+i)^{m+2}} \text{ \& so on.}$$

\therefore Present value ‘V’ of all instalments i.e–

$$= \frac{A}{(1+i)^{m+1}} + \frac{A}{(1+i)^{m+2}} + \dots\dots + \frac{A}{(1+i)^{m+n}}$$

$$\therefore V = \frac{A}{i} \frac{(1+i)^n - 1}{(1+i)^{m+n}} \dots\dots\dots(6) \text{ after simplification}$$

PRESENT VALUE OF DEFERRED PERPETUITY :

The formula is :

$$V = \frac{A}{i} \cdot \frac{1}{(1+i)^m} \dots\dots\dots(7)$$

Note:

Formula for Amount of deferred annuity will be :

$$M = \frac{A}{i} \left[\frac{(1+i)^n - 1}{(1+i)^m} \right]$$

N.B. : (i) All the above formula are based on yearly instalments of the instalments are paid half yearly, quarterly in monthly then replace 'i' by $\frac{i}{2}$ & n by 2n; i by $\frac{i}{4}$ & n by 4n and i by $\frac{i}{12}$ & n by 12 n respectively in all the formula upto (4) mentioned above.

(ii) An annuity of Rs. 500 means the value of each instalment is Rs. 500.

SOME EXAMPLES ON AMOUNT AND PRESENT VALUE :**TYPE I : TO FIND AMOUNT**

Example 1. Find the amount of an annuity of Rs.100 in 20 years allowing C.I. @ $4\frac{1}{2}\%$ pa. Given $\log 1.045 = 0.0191163$ and $\log 0.024117 = 2.3823260$.

Solution: It is a case of amount for immediate annuity as nothing is mentioned about the modality of payment of instalments, we shall consider that the instalments are paid at the end of each year.

Here, $A = \text{Rs. } 100$, $i = \frac{4\frac{1}{2}}{100} = 0.045$, $n = 20$, $M = ?$

$$\begin{aligned} \text{We known, } M &= \frac{A}{i} \left\{ (1+i)^n - 1 \right\} \\ &= \frac{100}{0.045} \left\{ (1.045)^{20} - 1 \right\}, \text{ Let } x = (1.045)^{20} \\ \therefore \log x &= 20 \log 1.045 \\ &= 20 \times 0.0191163 \\ &= 0.3823260 \end{aligned}$$

$$\begin{aligned} \text{Now, } M &= \frac{100}{0.045} (1.4117) \therefore x = \text{anti log } 0.3823260 \\ &= \text{Rs. } 3137.12 \\ &= 2.4117 \text{ [from given data, Note this step]} \end{aligned}$$

\therefore Required amount = Rs.3137.12.

Example 2. Find the amount of an annuity of Rs.800 payable at the end of each 3 months for 3 years @8% pa C.I. compounded quarterly.

Solution: Here, A=Rs.800, $i = \frac{8}{100} = 0.08$, int. compounded quarterly, n=3 years, M=?

$$\begin{aligned} \text{We know, } M &= \frac{A}{\frac{i}{4}} \left\{ \left(1 + \frac{i}{4} \right)^{4n} - 1 \right\} \\ &= \frac{800}{0.02} \left\{ (1.02)^{4 \times 3} - 1 \right\} && \text{Let } x = (1.02)^{12} \\ &= \frac{800}{0.02} \left\{ (1.02)^{12} - 1 \right\} && \therefore \log x = 12 \log 1.02 \\ &= \frac{80000}{2} (0.269) && = 12 \times 0.0086 \\ &= 40000 (0.269) && = 0.1032 \\ \therefore M &= \text{Rs. } 10760 && \therefore x = \text{anti log } 0.1032 \\ & && = 1.269 \\ \therefore \text{ Required amount} &= \text{Rs. } 10760. && \end{aligned}$$

TYPE II : TO FIND PRESENT VALUE

Example 3. Aman retires at the age of 60 years & his employer gives him a pension of Rs.12000 a year paid in half-yearly instalments for the rest of his life. Reckoning his expectation of life to be 13 years and that interest is at 4% pa, payable half yearly, what single sum is equivalent to his pension?

Solution: Here half yearly instalment = Rs.6000 = A

$$i = \frac{4}{100} = 0.04, \text{ int. payable half yearly}$$

$$n = 13 \text{ years, } V = ?$$

$$\begin{aligned} \text{We have, } V &= \frac{A}{\frac{i}{2}} \left\{ 1 - \left(1 + \frac{i}{2} \right)^{-2n} \right\} \\ &= \frac{6000}{0.02} \left\{ 1 - (1.02)^{-26} \right\} \\ &= 300000 \left\{ 1 - (1.02)^{-26} \right\} && \text{Let } x = (1.02)^{-26} \\ & && \therefore \log x = -26 \log 1.02 \end{aligned}$$

$$\begin{aligned}
 \therefore M &= 300000(1-0.5975) && = -20 \times 0.0086 \\
 &= 300000 \times 0.6025 && = -0.2236 \\
 &= \text{Rs. } 1,80,750 && = -1 + 1 - 0.2236 \\
 &&& = 1.7764 \\
 \therefore x &= \text{anti log } 1.7764 \\
 &= 0.5975
 \end{aligned}$$

\therefore Required amount = Rs.1,80,750.

Example 4. What sum will buy an annuity of Rs.60 payable for 4 years, the rate of int. being $3\frac{1}{2}\%$ pa?

Solution : Here, Price of an annuity is the present value of the annuity.
 Here, A = Rs.60, n = 4, i = 0.035, V = ?
 Apply, V formula for immediate annuity.
 Ans : Rs. 219.77.

Example 5. A man borrows Rs.10,000 on condition to repay with C.I. @5% pa by annual instalments of Rs.1000 each. In how many years will the debt be paid off?

Solution: Here, V = Rs.10,000, A = Rs.1000, i = 0.05, n = ?

(**Note:** Borrowing a sum is the present value of an annuity.)

$$\begin{aligned}
 \text{We have, } V &= \frac{A}{i} \{1 - (1+i)^{-n}\} \\
 \Rightarrow 10000 &= \frac{1000}{0.05} \{1 - (1.05)^{-n}\} \\
 \Rightarrow 10 \times 0.05 &= 1 - (1.05)^{-n} \\
 \therefore (1.05)^{-n} &= 1 - 0.5 = 0.5
 \end{aligned}$$

Taking log on both sides we get

$$\begin{aligned}
 -n \log 1.05 &= \log 0.5 = (-1 + 0.6990) = -0.3010 \\
 \Rightarrow n \times 0.0212 &= 0.3010 \\
 \therefore n &= \frac{3010}{212} = 14.2 \text{ years approx.}
 \end{aligned}$$

\therefore Required time = 14.2 years approx.

Example 6. Find the present value of an annuity of Rs.75 per year for 15 years deferred for 7 years hence at 5%pa.

Solution: Here, $m = 7$, $n = 15$, $A = \text{Rs.}75$, $i = \frac{5}{100} = 0.05$, $V = ?$

$$\text{We have, } V = \frac{A}{i} \cdot \frac{(1+i)^n - 1}{(1+i)^{m+n}}$$

$$\text{Now, } V = \frac{75}{0.05} \cdot \frac{(1.05)^{15} - 1}{(1.05)^{7+15}}$$

$$= \frac{75}{0.05} \cdot \frac{(1.05)^{15} - 1}{(1.05)^{22}}$$

$$= \frac{7500}{5} \times \frac{1.08}{2.927}$$

$$= \frac{75 \times 108000}{5 \times 2927}$$

$$\therefore V = \text{Rs.}553$$

$$\text{Let } x = (1.05)^{15}$$

$$\therefore \log x = 15 \log 1.05$$

$$= 15 \times 0.0012$$

$$= 0.3180$$

$$\therefore x = \text{anti log } 0.318$$

$$= 2.080$$

$$\text{Let } y = (1.05)^{22}$$

$$\therefore \log y = 22 \log 1.05$$

$$= 22 \times 0.0212$$

$$= 0.4664$$

$$\therefore y = \text{anti log } 0.446$$

$$= 2.927$$

Example 7. A man deposits Rs.1200 in a bank at the end of each year @5% pa C.I. what would be his deposit after 15 years?

Solution : Hint: Here, $A = \text{Rs.}1200$, $i = .05$, $n = 15$, $M = ?$
(Ans: Rs.256920)

Example 8. The cost of a computer is Rs.1,00,000 & its life is estimated to be 20 years. The computer is expected to cost more by 20% over its Present cost after 20 years. Find the sum to be invested every year at 5% pa C.I. for 20 years to replace this computer.

Solution: Here, Here, $M = \text{Rs. } 1,00,000 + 20\% \text{ of Rs. } 1,00,000$
 $= \text{Rs. } 1,20,000$

$$i = \frac{5}{100} = 0.05, n = 20;$$

$$\text{Use: } M = \frac{A}{i} \{(1+i)^n - 1\}$$

Ans : Rs. 3625.38.

Example 9. A man wants to purchase a car for Rs.1,00,000. The contract is that he will pay Rs.60,000 cash & the balance in 20 equal annual instalments with 5% C.I. pa. Find the amount to be paid annually. (Given: $(1.05)^{-20} = 0.3767$)

Solution: Here, $V = (1,00,000 - 60,000)$
 $= \text{Rs. } 40,000$

$$n = 20, i = .05, A = ?$$

$$\text{Apply: } V = \frac{A}{i} \{1 - (1+i)^{-n}\}$$

Ans: Rs. 3208.73.

Example 10. A machine costs a company Rs.52,000 & its effective life is estimated to be 25 years. A sinking fund is created for replacing the machine by a new model at the end of its life time, when its scrap realises a sum of Rs.2500 only. The price of the new model is estimated to be 25% more than the price of the present one. What amount should be retained out of profit at the end of each year for the sinking fund, if it accumulates 3.5% pa CI.

Solution: Hint: $M = \text{Rs. } (52,000 + 25\% \text{ of } 52,000) - \text{Rs. } 2500$
 $= \text{Rs. } 40,000$

$$i = 0.035, n = 25, A = ?$$

$$\text{Use: } M = \frac{A}{i} \{(1+i)^n - 1\}$$

Ans: Rs. 1610.82.

Example 11. A firm anticipates a capital expenditure of Rs.5,00,000 for a new equipment in 10 years. For this purpose they kept aside Rs.35,000 out of profit at the end of each year. Find the surplus amount (if any) after incurring the expenditure if int. is accumulated at 8% pa CI.

Solution: Hint: Anticipated Expenditure = Rs. 5,00,000 (desired)

$$n = 10, i = \frac{8}{100} = 0.08, A = \text{Rs. } 35,000$$

Let us find the amount of annuity of Rs.35,000 for 10 years @8% pa CI for 10 years.

$$\begin{aligned} M &= \frac{A}{i} \left\{ (1+i)^n - 1 \right\} \\ &= \frac{35000}{0.08} \left\{ (1.08)^{10} - 1 \right\} \\ &= \frac{35000}{8} (1.158) & \text{Let } x &= (1.08)^{10} \\ &= \frac{3500 \times 1158}{8} & \therefore \log x &= 10 \log 1.08 \\ &= \text{Rs. } 5,06,525 & &= 10 \times 0.0334 \\ & & &= 0.334 \\ & & \therefore x &= \text{anti log } 0.334 \\ & & &= 2.158 \end{aligned}$$

Thus we find that after incurring the expenditure of Rs. 5,00,000, surplus will be = Rs.6625.

EXERCISE

1. Calculate the present value of an annuity of Rs.1200 per annum for 12 years at 8% pa CI.
(Ans: Rs.9036)
2. Find the amount of an immediate annuity of Rs.100 pa left unpaid for 10 years allowing 5% pa CI.
(Ans: Rs.1258)
3. What sum will buy an annuity of Rs.1050 for 4 years, the rate of int being 3.5% pa CI?
(Ans: Rs.3846)
4. A man decides to deposit Rs.5000 at the end of each year in a bank at 5% pa CI. What will be his total accumulation at the end of 15 years?
(Ans: Rs.1,08,000)

5. A limited company intends to create a depreciation fund to replace, at the end of 25th year, assets costing Rs.1,00,000. Calculate the amount to be retained out of the profit's every year if int. rate is 3% pa.
(Ans: Rs.2,755)
6. A machine costs the company Rs.97,000 & its effective life is estimated to be 12 years. A fund is created for replacing the machine at the end of its effective life time. If scrap realises Rs.2000 only, what amount should be retained out of the profit at the end of each year to accumulate at 5% pa CI.
(Ans: Rs.5960 approx)
7. A man borrows Rs.3000 at 6% pa CI and promises to pay off the loan in 20 annual payments beginning at the end of first year. What is the annual payment necessary?
(Ans: Rs.261.57)
8. A person wishes to buy a printing machinery valued at Rs.17,000. He is prepared to pay now Rs.9000 & the balance in 8 equal annual instalments at $3\frac{1}{2}\%$ pa CI. How much should he pay annually?
(Ans: Rs.1167 approx)
9. A machine is bought on instalment basis, such that Rs.5,000 is to be paid on signing the contract and four yearly instalments of Rs.3000 each payable, at the end of 1st, 2nd, third and fourth year. If int. cost is charged at 5% pa CI, what should be the cash down price?
(Ans: Rs.15,644 approx)
10. A man deposits his whole fortune Rs.20,000 in a bank at 5% pa CI & settles to withdraw Rs.1800 per year for this personal expenses. If he begins to spend from the end of the first year and goes on spending at this rate, show that he will be ruined before the end of 17th year.
11. A man borrowed some money and paid back in 3 equal annual instalments of Rs.21,600 each. What sum did he borrow, if int. charged is 20% pa CI? Find the interest charged.
(Ans: Rs.45511.20 & Rs.19288.80)
12. What sum should be invested every year at 8% pa CI for 10 years to replace plant and machinery which is expected to cost then 20% more than its present cost of Rs.50,000?
(Ans: Rs.4142)

UNIT-2

LINEAR INEQUALITIES & THEIR GRAPHS

Let a and b be two real numbers. Then a is greater than b if $a-b$ is positive. In symbol, $a > b$ if $a-b > 0$. For example, if $a = 2$, $b = -1$ then $a-b = 2-(-1) = 3 > 0$. So $a > b$ i.e. $2 > -1$.

Similarly, a is less than (or smaller than) b if $a-b$ is negative, i.e. $a < b$ if $a-b < 0$. For example, if $a = -5$, $b = -3$, then $a-b = -5-(-3) = -2 < 0$

$\therefore a < b$ i.e. $-5 < -3$.

Further $a \geq b$ means either $a = b$ or $a > b$ and $a \leq b$ means either $a = b$ or $a < b$.

On the number line, for any two real numbers a and b , $a > b$ means a line on the right of b and $a < b$ means a line on the left of b .

There are two types of inequalities (i) Absolute inequality and (ii) Conditional inequality.

If an inequality is true for all values of the variable, then the inequality is said to be an **absolute inequality**. For example, if $x \in \mathbb{R}$ then $x^2 \geq 0$ is an absolute inequality. Because we know that for any real number x , $x^2 \geq 0$. But the inequality like $5x > 7$ is a conditional inequality, as we see that $5x > 7$ if $x > \frac{7}{5}$. So this inequality is true for real numbers greater than $\frac{7}{5}$.

SOME PROPERTIES OF INEQUALITIES :

If a, b, c are real numbers, then

- (i) $a > b$ and $b > c \Rightarrow a > c$
- (ii) $a > b \Rightarrow a+c > b+c$
- (iii) $a > b$ and $c > 0 \Rightarrow ac > bc$
- (iv) $a > b$ and $c < 0 \Rightarrow ac < bc$

Proof : (i) We have $a-c = (a-b)+(b-c)$

Since $a > b$, $b > c$, therefore $a-b > 0$, $b-c > 0$

$\therefore (a-b)+(b-c) > 0$

Thus $a-c > 0$

$\therefore a > c$

(ii) $(a+c) - (b+c) = a-b$

> 0 [$\because a > b$]

$\therefore a+c > b+c$

(iii) $ac-bc = (a-b)c$

> 0 [$\because a > b \Rightarrow a-b > 0$ also $c > 0$ given]

$\therefore ac > bc$

(iv) $ac-bc = (a-b)c$

< 0 [$\because a > b \Rightarrow a-b > 0$ but $c < 0$ given]

$\therefore ac < bc$

LINEAR EXPRESSIONS, LINEAR EQUATIONS AND LINEAR INEQUATIONS

An expression of the type $ax+b$ ($a \neq 0$) is said to be a **linear expression** in one variable. A linear expression in two variables is of the form $ax+by+c$ ($a \neq 0, b \neq 0$). A linear expression in three variables is of the form $ax+by+cz+d$ ($a \neq 0, b \neq 0, c \neq 0$) and so on.

When a linear expression is equated to a constant (usually the number zero), then it is said to be a linear equation.

A linear equation in two variables x and y always represents a straight line. Thus the graph of the equation $ax+by+c=0$ (where a and b are not simultaneously zero) is a straight line.

An inequality of the type $ax+b > 0$ or $ax+b < 0$ or $ax+b \geq 0$ or $ax+b \leq 0$ ($a \neq 0$) is a **linear inequality (or a linear inequation)** in one variable.

An inequality of the type $ax+by+c > 0$ or $ax+by+c < 0$ or $ax+by+c \geq 0$ or $ax+by+c \leq 0$ (where a and b are not simultaneously zero) is called a **linear inequality (or a linear inequation)** in two variables.

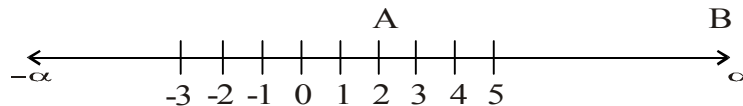
GRAPHICAL REPRESENTATION OF LINEAR INEQUATITIES :

When we are to draw the graph of a linear inequality in one variable, we use the number line. This is illustrated with the help of the following examples.

Example 1: Draw the graphs of the following inequations in one variable :

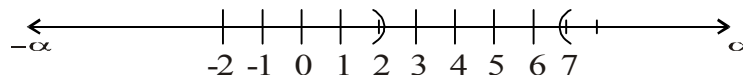
(i) $x \geq 2$ (ii) $2 < x < 7$ (iii) $2x+1 > 0$

Solution : (i) Since $x \geq 2$, so x may take the value equal to 2 and any value greater than 2. So every point on the number line which lies on the right of 2 (including 2) satisfies the inequation $x \geq 2$.



Hence the required graph is the ray \overrightarrow{AB} (the shaded portion of the number line)

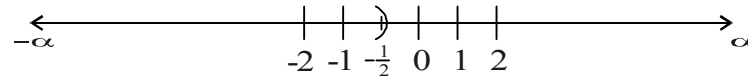
(ii) Here $x > 2$ and $x < 7$. So all values greater than 2 but less than 7 will satisfy the given inequation.



So the graph of the given inequation is the shaded portion of the number line (excluding the points 2 and 7)

(iii) $2x+1 > 0$
 $\Rightarrow 2x > -1$
 $\Rightarrow x > -\frac{1}{2}$

So all values greater than $-\frac{1}{2}$ will satisfy the given inequation.

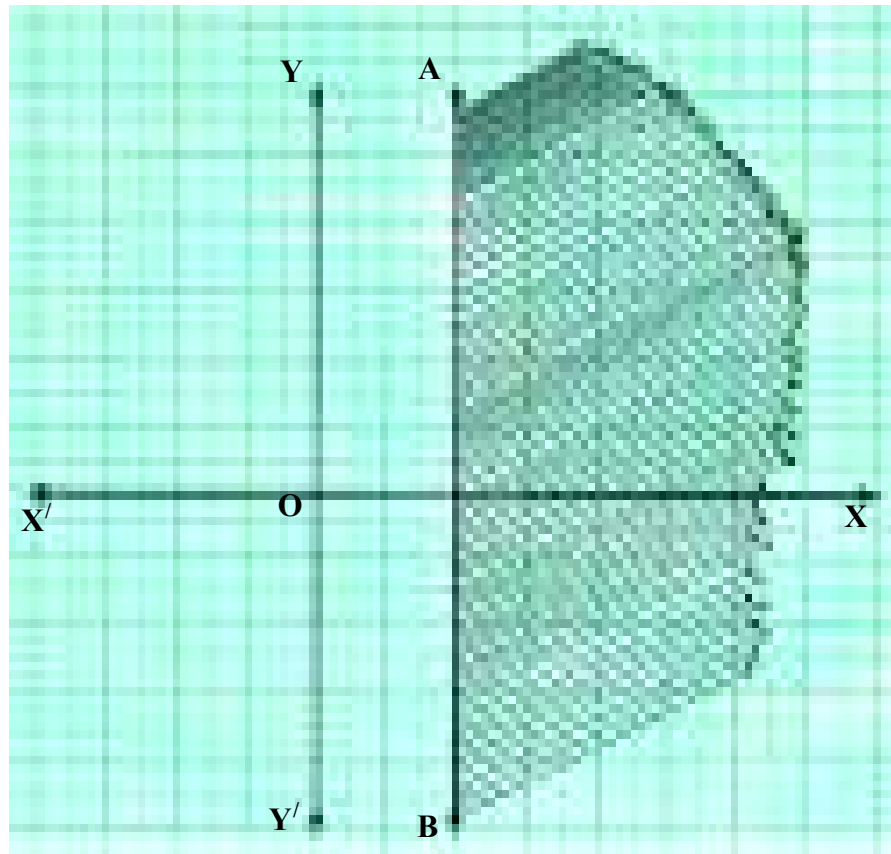


Hence the graph of the given inequation is the shaded portion (excluding the point $-\frac{1}{2}$) of the number line.

However we may consider an inequation in one variable as an inequation in two variables also. For example, if we consider the inequation $x \geq 2$ as a linear inequation in two variables x and y then it will mean that $x \geq 2$ and y can take any value (positive, negative and zero).

Example 2: Let us draw the graph of the inequation $x \geq 2$ in the coordinate plane (i.e. xy -plane).

Solution : First we shall draw the graph of the equation $x=2$ in the xy -plane. The line AB parallel to the y -axis at a distance of 2 units from the y -axis will represent the graph of the equation $x=2$.



Scale: 1 big division=1 unit

The line AB divides the coordinate plane into two parts. For all points to the left of AB, $x < 2$; for all points to the right of AB, $x > 2$ and for all points on the line AB, $x = 2$.

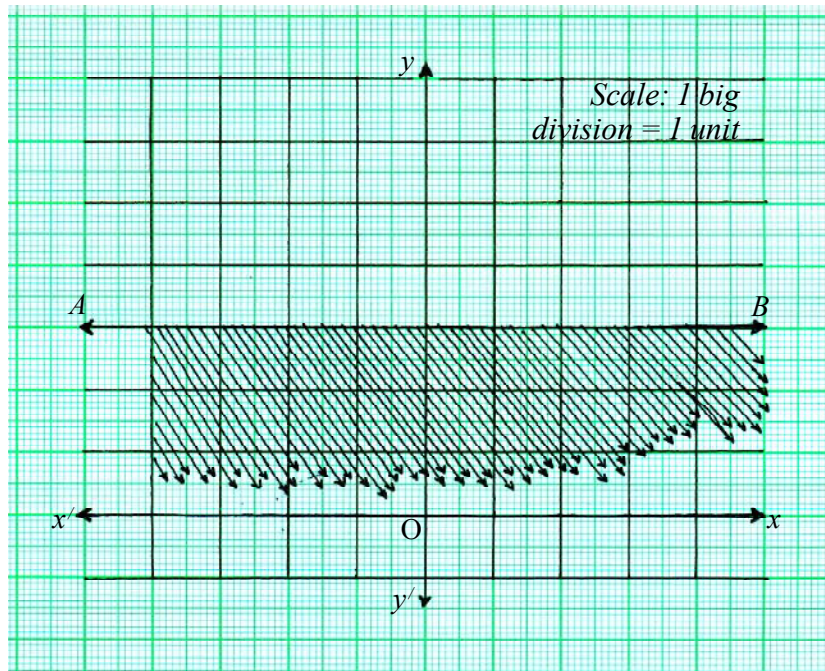
Hence the graph of the inequation $x \geq 2$ is the shaded portion in the xy -plane to the right of the line AB including the line AB.

Note:

The graph of $x > 2$ is the shaded portion to the right of the line AB excluding the line AB.

Example 3: Draw the graph of $y < 3$ in the xy -plane.

Solution: We shall first draw the graph of the equation $y = 3$ in the xy -plane.



The graph of $y = 3$ is the line AB parallel to x -axis at a distance of 3 units from the x -axis.

Now, above the line AB, $y > 3$ and below the line AB, $y < 3$ but on the line AB, $y = 3$.

Hence the graph of $y < 3$ is the shaded space below the line AB, excluding the line AB.

Example 4: Draw the graph of $x > y$.

Solution: First we shall draw the graph of $x = y$ (i)

Some pairs of values of x and y satisfying—

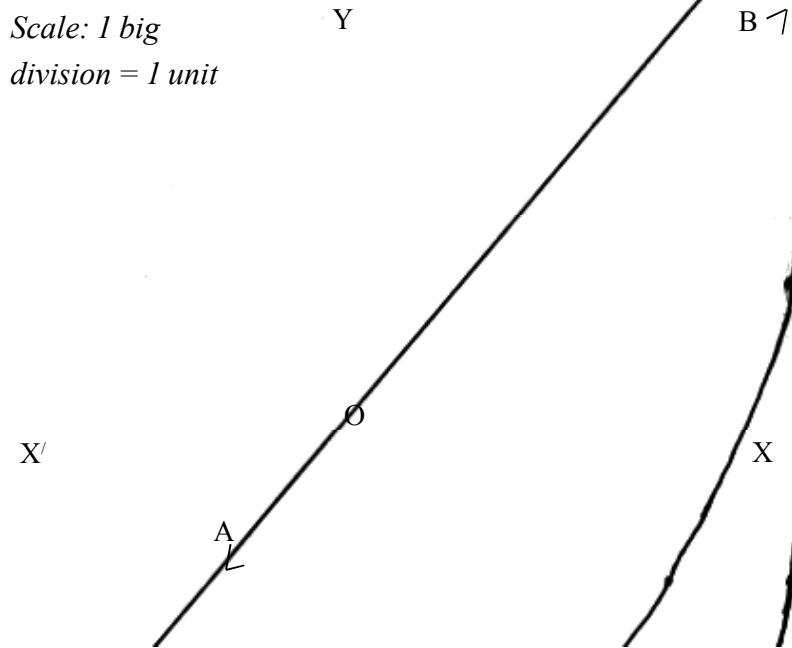
(i) are as follows :

x	0	1	2
y	0	1	2

Plotting these points on the graph paper and joining them we get the line AB as the graph of $x = y$.

Now the line AB divided the xy -plane into two parts. In the shaded portion, for all points $x > y$. But on the other side of the line AB, for all points $x < y$.

Hence the graph of the given inequation is the shaded space (excluding the line AB) as shown in the figure.



Example 5: Draw the graph of the inequation $x+3y-2 < 0$

Solution : First we shall draw the graph of the equation $x+3y-2 = 0$ (i)

From (i) $y = \frac{2-x}{3}$

Some pairs of values of x and y satisfying (i) are as follows :

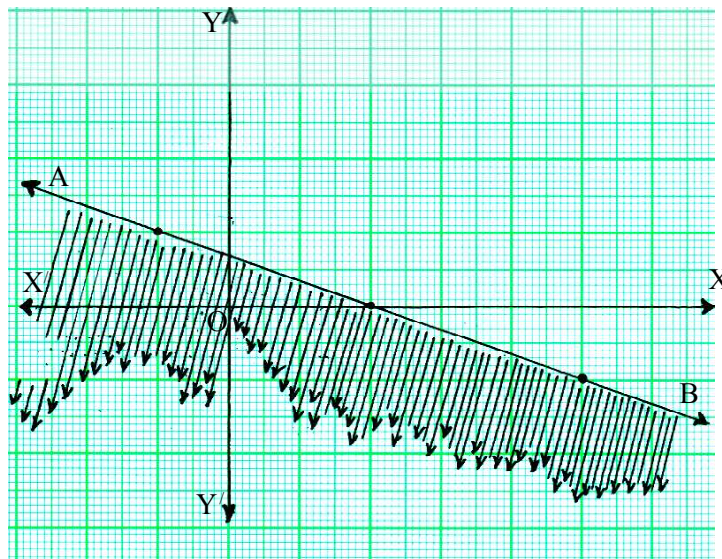
x	2	-1	5
y	0	1	-1

Plotting these points on a graph paper and joining them we get the line AB as the graph of the equation (i)

Now,

$$x+3y-2 < 0 \Rightarrow y < \frac{2-x}{3}$$

We see that for all points above the line AB,



Scale : 1 big division = 1 unit

$$y > \frac{2-x}{3} \text{ and for all points below AB, } y < \frac{2-x}{3}.$$

Hence the graph of the given inequation is the shaded space below the line \overline{AB} , excluding the line \overline{AB} .

Example 6 : Solve graphically :

$$3x+y=6$$

$$2x+3y-12=0$$

Solution : First we shall draw the graph of the equations

$$3x+y=6 \quad \dots\dots\dots (i)$$

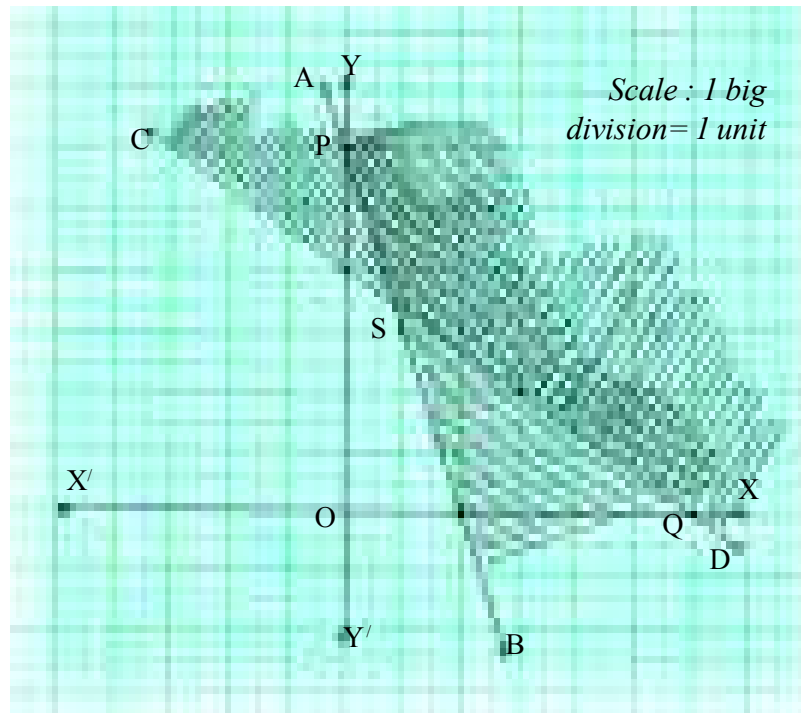
$$2x+3y-12=0 \quad \dots\dots\dots (ii)$$

From (i) $y=6-3x$

Some pairs of values of x and y satisfying (i) are

x	0	2	1
y	6	0	3

Plotting these points on a graph paper and joining them we get the line AB as the graph of the equation (i)



Now, $3x+y > 6$

$$\Rightarrow y > 6-3x$$

For all points to the right of AB, $y > 6-3x$.

Hence the graph of the first inequation is all the points to the right of the line AB (excluding the line AB).

Again from (ii) $y = \frac{12-2x}{3}$

Some pairs of values of x and y satisfying (ii) are

x	0	6	3
y	4	0	2

Plotting these points on the same graph paper (with the same scale) we get the line CD as the graph of the equation (ii)

Now, $2x+3y-12 > 0 \Rightarrow y > \frac{12-2x}{3}$.

Therefore the graph of the second inequation is all the points above the line CD (excluding the line CD).

The common portion of the graphs of two inequation lies within the unbounded region APSQD as shown in the figure by double crossing lines (excluding the rays \overrightarrow{SA} and \overrightarrow{SD}).

Hence this common space is the graphical solution of the two inequations.

EXERCISE

1. Define a linear inequation in one variable.
2. Define a linear inequation in two variables.
3. What do you mean by (i) absolute inequality and (ii) conditional inequality?
4. Draw the graphs of the following inequations considering as linear inequality in one variable :
 - (i) $x > -2$ (ii) $x \leq 0$ (iii) $x > 0$
 - (iv) $|x-1| < 3$ (v) $3 < x < 8$
5. Draw the graphs of the following inequations in the xy -plane :
 - (i) $x \geq 5$ (ii) $y < 3$ (iii) $x \geq 0$
 - (iv) $\frac{x}{3} + \frac{y}{4} > 1$ (v) $2x-4y > 5$
 - (vi) $x+2y < 6$ (vii) $2x+3y > 12$
 - (viii) $x < y$ (ix) $x+y < 2$
6. Solve graphically the following system of inequations :
 - (i) $2x+y \geq 12, x+y \geq 7$
 - (ii) $2x+5y < 10, x < 2$
 - (iii) $2x+3y \geq 18, x+y \geq 8$
 - (iv) $y \geq 5x, y \leq 0$
 - (v) $\frac{x}{2} + \frac{y}{3} > 6, \frac{x}{2} - \frac{y}{3} \leq 1$

PERMUTATION & COMBINATION

Choices : Suppose there are two doors (A and B) in a room. If you want to enter into the room, you may enter through the door A or through the door B. So there are two choices for the event of entering into the room.

After entering into the room if you are to come out of the room, then for this event also, there are two choices i.e. you may choose the door A or the door B to come out of the room. We may show the choices of entering into and leaving out the room as follows :

	enter	leave
door name:	A	A
	A	B
	B	A
	B	B

Thus there are four different ways of entering into and leaving out of the room.

But if there is a restriction that you cannot come out of the room by the same door through which you have entered into the room, then you have only one choice to come out of the room. We may tabulate the choices of entering into and leaving out as follows :

enter	leave
A	B
B	A

Thus there are two choices of entering into and leaving the room.

Similarly, suppose there are three doors, A, B, C in a room. You can enter into the room in three different ways. After entering into the room if you come out by a different door, then you have two different ways for coming out of the room. We may tabulate the possibilities of entering into by one door and leaving out by a different door as follows :

enter	leave
A	B
A	C
B	A
B	C
C	A
C	B

Thus there are six different ways of entering into and leaving out the room.

We see that the first operation i.e. the entering into can be performed by 3 different ways and corresponding to each of these performance the second operation i.e. to leave the room can be performed in 2 different ways. Thus the total number of ways that both the operations (entering into and leaving out) can be performed in 3×2 i.e. 6 different ways.

If the restriction of coming out through the same door by which you have entered is removed, then the operation of entering into can be performed in 3 different ways and for each performance, the second operation i.e. leaving the room can be performed in 3 different ways. So both the operations can be performed in 3×3 i.e. 9 different ways.

From the above discussion it is clear that there is a basic principle which may be stated as follows :

Principle : If one operation can be performed in 'm' different ways and corresponding to each performance of this operation another operation can be performed in 'n' different ways, then both the operations can be performed simultaneously in $m \times n$ different ways.

The principle can be extended to any number of operations. In general if one operation can be performed in 'm' different ways and if a second operation can be performed in 'n' different ways and a third can be performed in 'p' different ways and so on, then all the operations can be performed simultaneously in $m \times n \times p \times \dots$ different ways.

Permutation : The different arrangements in a row which can be made by taking some of the objects or all the objects of a given finite set are called permutations.

For example, if there are three letters A, B, C, then permutations of these letters taking one, two or three at a time are as follows :

one at a time	two at a time	three at a time
A	AB	ABC
B	BA	ACB
C	AC	BAC
	CA	BCA
	BC	CAB
	CB	CBA

Notation : Let the number of objects in a given set be 'n' and the number of objects in each permutation be 'r' ($r \leq n$). Then the number of permutations is denoted by ${}^n P_r$ or ${}_n P_r$ or $P(n, r)$. Thus the notation ${}^n P_r$ or ${}_n P_r$ or $P(n, r)$ will mean the number of permutations of 'n' different things taking 'r' at a time. Thus in the above example, we see that ${}^3 P_1 = 3$, ${}^3 P_2 = 6$, ${}^3 P_3 = 6$.

Factorial Notation : The continued product of the first n natural numbers is denoted by the symbol \underline{n} or n! and is read as “Factorial n”.

$$\begin{aligned} \text{Thus } \underline{n} \text{ or } n! &= 1 \times 2 \times 3 \times \dots \times (n-1) \times n \\ &= n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1 \end{aligned}$$

For example, $\underline{5} = 1 \times 2 \times 3 \times 4 \times 5 = 5 \times 4 \times 3 \times 2 \times 1$

$$\underline{7} = 1 \times 3 \times 4 \times 5 \times 6 \times 7 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

From the definition, we see that

$$\begin{aligned} \underline{n} &= n (n-1)(n-2) \dots \dots \dots 3.2.1 \\ &= n [(n-1)(n-2) \dots \dots \dots 3.2.1] \\ &= n \underline{n-1} \end{aligned}$$

Similarly, $\underline{n} = n(n-1) (n-2) = n (n-1)(n-2) \underline{n-3}$
and so on

Thus $\underline{10} = 10 \times \underline{9} = 10 \times 9 \times \underline{8} = 10 \times 9 \times 8 \times \underline{7}$ and so on.

Theorem : The number of permutations of n different things taken r (r ≤ n) at

a time is $\frac{\underline{n}}{\underline{n-r}}$

Proof : The number of permutations of n different things taken r at a time is the same as the number of arrangements of r things taken from the given n different things in r different places in a row.

Now the first place can be filled up by any one of the given n different things. So the first place can be filled up by n different ways. After the first place is filled up, the second place can be filled up by any one of the remaining (n-1) different things. So the second place can be filled up by (n-1) different ways. Similarly the third place can be filled up by (n-2) different ways and so on. Continuing this process, we see that for the last place (i.e. the rth place) there remain [n-(r-1)] different things. So the last place can be filled up in [n-(r-1)]=(n-r+1) different ways.

Thus the r places can be filled up in n(n-1)(n-2) (n-r+1) different ways.

$$\begin{aligned} \text{Hence } {}^n P_r &= n(n-1)(n-2) \dots \dots \dots (n-r+1) \\ &= \frac{n(n-1)(n-2) \dots \dots \dots (n-r+1)(n-r) \dots \dots \dots 3.2.1}{(n-r)(n-r+1) \dots \dots \dots 3.2.1} \\ &= \frac{\underline{n}}{\underline{n-r}} \end{aligned}$$

Corollary : (1) Putting $r = n$ in ${}^n P_r = n(n-1)(n-2)\dots(n-r+1)$

We get ${}^n P_n = n(n-1)(n-2)\dots\dots\dots 1 = \underline{n}$

(2) Again putting $r = n$ in ${}^n P_r = \frac{\underline{n}}{\underline{n-r}}$,

We get ${}^n P_n = \frac{\underline{n}}{\underline{0}}$

$$\therefore \underline{n} = \frac{\underline{n}}{\underline{0}}$$

So, we may define $\underline{0} = 1$.

Example 1 : Find the values of ${}^5 P_3$, ${}^7 P_4$.

Solution : ${}^5 P_3 = \frac{\underline{5}}{\underline{5-3}} = \frac{\underline{5}}{\underline{2}} = \frac{5 \times 4 \times 3}{\underline{2}} = 60$

$${}^7 P_4 = \frac{\underline{7}}{\underline{7-4}} = \frac{\underline{7}}{\underline{3}} = \frac{7 \times 6 \times 5 \times 4}{\underline{3}} = 840$$

Example 2 : In how many ways can the letters of the word 'EQUATION' be arranged?

Solution : There are 8 different letters in the word EQUATION. Therefore number of ways of arrangements is the permutations of 8 different things taken all at a time.

$$= {}^8 P_8 = \underline{8} = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

Permutations with repetition : To find the number of permutations of n different things taken r at a time, when each thing may be repeated upto r times.

Clearly the required number of permutations is the same as the number of ways in which r places can be filled up with n different things when each of the n things can be repeated upto r times in any permutation.

The first place can be filled up by placing any one of the n things i.e. the first place can be filled up in n different ways. Since the things may be repeated, corresponding to each way of filling up of the first place, the second place can be filled up by any one of the given n different things in n different ways. Similarly the third place can also be filled up in n different ways. Thus each of the r places can be filled up in n different ways. Hence the total number of ways of filling up the r places is $n.n.n\dots\dots(r \text{ times}) = n^r$.

Thus the required number of permutations is equal to n^r .

Example 3 : In how many ways can 6 letters be posted in 5 letter boxes?

Solution : The first letter can be posted in any one of the 5 boxes. When this has been done, the second letter can also be posted in any one of the 5 boxes (the second letter can be posted in the box in which the first one is posted). Similarly each of the 6 letters can be posted in 5 different ways.

Hence the required number of ways

$$\begin{aligned} &= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \\ &= 5^6 \\ &= 15625. \end{aligned}$$

Example 4 : In how many ways can 5 prizes be awarded to 4 boys if each boy is eligible for any number of prizes?

Solution : First prize can be awarded to any one of the 4 boys. So the first prize can be awarded in 4 different ways. After giving the first prize, the second one can also be awarded to any one of the 4 boys. So the second prize can be awarded in 4 different ways. Thus each of the 5 prizes can be awarded in 4 different ways.

Hence required number of ways

$$\begin{aligned} &= 4 \times 4 \times 4 \times 4 \times 4 \\ &= 4^5 \\ &= 1024. \end{aligned}$$

PERMUTATIONS OF THINGS NOT ALL DIFFERENT :

To find the number of permutations of n things taken all at a time, when the things are not all different.

Let a, b, c, \dots are the n things and suppose that 'p' of them are a's, 'q' of them are b's and 'r' of them are c's and the rest are all different. To find the number of permutations of these things taken all at a time.

Let the total number of permutations be x . If all the a's are replaced by p new distinct things, then in each of x permutations, these p new things can be arranged among themselves in $|p|$ ways. So we get $x \cdot |p|$ permutations from the original x permutations.

In each of these $x \cdot |p|$ permutations there will be q like things all b's and r like things all c's. If we replace all the b's by q new distinct things different from the rest, then these q distinct things can be arranged among themselves in $|q|$ ways in each of the $x \cdot |p|$ permutations. So from $x \cdot |p|$ permutations we now get $x \cdot |p| \cdot |q|$ permutations each containing r like things all c's.

As earlier by replacing all the c's by r new distinct thing different from the rest, we get

$\times \underline{p} \underline{q} \underline{r}$ permutations each containing n different things. But from n different things taken all at a time we get \underline{n} permutations.

$$\therefore x = \underline{p} \underline{q} \underline{r} = \underline{n}$$

$$\Rightarrow x = \frac{\underline{n}}{\underline{p} \underline{q} \underline{r}}$$

Example 5 : Find the number of ways in which the letters of the word 'COLLEGE' be arranged.

Solution : Here total number of letters = 7
Out of 7 letters 2 are L's and 2 are E's

$$\begin{aligned} \therefore \text{Total number of arrangements} &= \frac{\underline{7}}{\underline{2} \underline{2}} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3}{2} \\ &= 1260 \end{aligned}$$

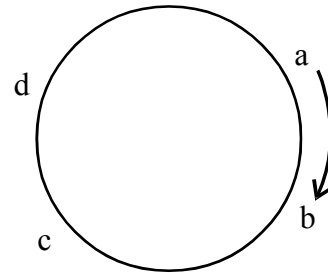
CIRCULAR PERMUTATIONS :

In case of linear permutations (i.e. arrangement in a row), there are two ends but in case of circular permutation (i.e. arrangement in a circle) there is neither a beginning nor an end. For example, let us consider the circular arrangement of the 4 letters a, b, c, d as shown in the figure

Starting from each letter we can read it in clockwise direction in any one of the following ways :

a b c d, b c d a, c d a b, d a b c.

If we arrange in a row these four will give 4 different arrangements. Thus corresponding to 4 linear permutations we get only one circular permutation. But 4 letters can be arranged in a row in $\underline{4}$ different ways.



$$\therefore \text{Number of permutations of 4 letters arranging in a circle} = \frac{\underline{4}}{4} = \underline{3} = 6$$

TO FIND THE NUMBER OF PERMUTATIONS OF N DIFFERENT THINGS IN CIRCULAR ARRANGEMENT :

Let the n different things be represented by the letters a_1, a_2, \dots, a_n

If we arrange in a row then these n different things can be arranged in \underline{n} ways. In a

row the arrangements $a_1 a_2 a_3 \dots a_n, a_2 a_3 a_4 \dots a_n a_1, a_3 a_4 a_5 \dots a_n a_1 a_2, \dots, a_n a_1 a_2, \dots, a_{n-1}$ are n different arrangements but in a circle all these n arrangements are same. Thus to every n different arrangement in a row there is only one arrangement in a circle.

$$\begin{aligned} \therefore \text{Total number of arrangements in a circle} &= \frac{n}{n} \\ &= \underline{n-1} \end{aligned}$$

Example 6 : In how many ways can 7 persons be seated around a round table?

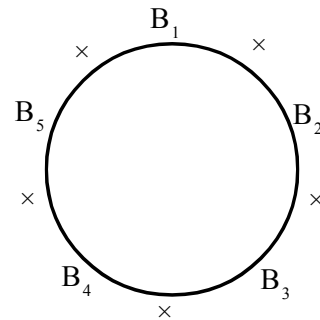
Solution : 7 persons can sit around a round table in $7-1 = \underline{6} = 720$ ways.

Example 7 : There are 5 boys and 4 girls to sit in a circle. In how many ways can they sit?

Solution : Total number of persons = $5+4=9$
So 9 persons are to be seated in a circle.
 \therefore Total number of ways = $9-1 = \underline{8}$

Example 8 : In how many ways can 5 boys and 4 girls be arranged in a circle so that no two girls are together?

Solution : Let the boys be arranged first. They can be arranged in a circle in $\underline{5-1} = \underline{4}$ ways. Since no two girls are together therefore when the boys have occupied their seats in any one of these $\underline{4}$ arrangements, the 4 girls are to occupy any 4 seats out of 5 seats (see figure) one between every two boys. This can be done in 5P_4 ways.



\therefore Required number of arrangements

$$\begin{aligned} &= \underline{4} \times {}^5P_4 \\ &= \underline{4} \times \dots\dots \\ &= \underline{4} \times \underline{5} \\ &= 2880 \end{aligned}$$

Example 9 : In how many ways can 6 different beads be placed in a necklace?

Solution : Here we are to form circular permutations of 6 different things taken all at a time. But in case of necklace, arrangement counting from clockwise and anticlockwise directions are not different.

Hence the total number of way of arrangements
 $= \frac{1}{2} |6-1| = \frac{1}{2} |5| = \frac{1}{2} \times 120 = 60$

RESTRICTED PERMUTATIONS :

Let us give some illustrative example in which there are some restriction in the arrangements.

Example 10 : In how many ways can the letters of the word 'DAUGHTER' be arranged so that the letters G, T and R never occur?

Solution : Since the letters G, T and R never occur in the arrangements so we are to arrange the remaining 5 letters among themselves. This can be done in $|5|$ ways.

Required number of arrangements = $|5| = 120$.

Example 11 : In how many ways can the letters of the word 'DAUGHTER' be arranged taking 5 at a time in which the letters G, T, R always occupy the first, third and fifth positions respectively.

Solution : Put the three letters G, T, R in the specified positions (1st, 3rd and 5th respectively). Then we are to fill up the remaining 2 places (2nd and 4th) with the remaining 5 letters which can be done in 5P_2 ways.

\therefore Required number of arrangements = ${}^5P_2 = 20$.

Example 12 : Find the number of ways in which 8 boys be seated in 6 chairs in a row so that 3 particular boys are always included.

Solution : Three particular boys can occupy 3 chairs out of 6 in 6P_3 ways. After filling up the 3 chairs, the remaining $(6-3) = 3$ chairs can be occupied by the remaining $(8-3) = 5$ boys in 5P_3 ways.

Hence required number of ways

$$= {}^6P_3 \times {}^5P_3$$

$$= \frac{|6|}{|3|} \times \frac{|5|}{|2|}$$

$$= 7200.$$

Example13 : In how many ways can the letters of the word TABLE be arranged taken 4 at a time so that BL is always together and also in the given order?

Solution : Let us consider BL together in that order as a single letter which can occupy 2 of the 4 places. Now from the remaining 3 letters (T, A, E) taking 2 at a time can be arranged in 3P_2 ways. In each of these 3P_2 permutations there are 2 letters. In between then there is a 1 place and at the two ends there are 2 places; so in total there are 3 places which can be occupied by the group of letters BL in that order by 3 different ways.

$$\begin{aligned} \text{Hence the total number of ways} &= 3 \times {}^3P_2 \\ &= 18 \end{aligned}$$

Note :

If however BL may not be in that order, then number of ways = $3 \times \underline{2} \times {}^3P_2 = 36$
(In this case the letters B and L can be arranged among themselves in ${}^2P_2 = \underline{2}$ ways).

WORKED OUT EXAMPLES

1. Prove that ${}^n P_{n-1} = {}^n P_n$

Solution : We have ${}^n P_r = \frac{\underline{n}}{\underline{n-r}}$

$$\therefore {}^n P_{n-1} = \frac{\underline{n}}{\underline{(n-(n-1))}} = \frac{\underline{n}}{\underline{1}} = \underline{n} = {}^n P_n$$

$$\therefore {}^n P_{n-1} = {}^n P_n$$

2. Prove that ${}^n P_r = n \cdot {}^{n-1} P_{r-1}$

Solution : ${}^n P_r = \frac{\underline{n}}{\underline{n-r}}$

$$= \frac{n \underline{n-1}}{\underline{(n-1)-(r-1)}}$$

$$= n \cdot \frac{\underline{n-1}}{\underline{(n-1)-(r-1)}}$$

$$= n \cdot {}^{n-1} P_{r-1}$$

3. Find the value of n if ${}^n P_5 : {}^n P_3 = 2:1$

Solution : $\frac{{}^n P_5}{{}^n P_3} = \frac{2}{1}$

$$\Rightarrow {}^n P_5 = 2 \times {}^n P_3$$

$$\Rightarrow \frac{|n|}{|n-5|} = 2 \times \frac{|n|}{|n-3|}$$

$$\Rightarrow \frac{|n-3|}{|n-5|} = 2$$

$$\Rightarrow \frac{(n-3)(n-4)|n-5|}{|n-5|} = 2$$

$$\Rightarrow (n-3)(n-4) = 2 \times 1$$

$$\Rightarrow n-3 = 2$$

$$\Rightarrow n = 5$$

4. Evaluate n if ${}^n P_4 : {}^{n+1} P_4 = 5:9$

Solution : $\frac{{}^n P_4}{{}^{n+1} P_4} = \frac{5}{9}$

$$\Rightarrow \frac{\frac{|n|}{|n-4|}}{\frac{|n+1|}{|n+1-4|}} = \frac{5}{9}$$

$$\Rightarrow \frac{|n| |n-3|}{|n+1| |n-4|} = \frac{5}{9}$$

$$\Rightarrow \frac{|n| (n-3) |n-4|}{(n+1) |n| |n-4|} = \frac{5}{9}$$

$$\begin{aligned} \Rightarrow \frac{n-3}{n+1} &= \frac{5}{9} \\ \Rightarrow 9n-27 &= 5n+5 \\ \Rightarrow 4n &= 32 \\ \Rightarrow n &= 8. \end{aligned}$$

5. If ${}^{n+1}P_3 = 10 \times {}^{n-1}P_2$, find n.

Solution : ${}^{n+1}P_3 = 10 \times {}^{n-1}P_2$

$$\begin{aligned} \Rightarrow \frac{|n+1|}{|n+1-3|} &= 10 \times \frac{|n-1|}{|n-1-2|} \\ \Rightarrow \frac{(n+1)n|n-1|}{|n-2|} &= 10 \times \frac{|n-1|}{|n-3|} \\ \Rightarrow \frac{n(n+1)}{(n-2)|n-3|} &= 10 \times \frac{1}{|n-3|} \\ \Rightarrow n(n+1) &= 10(n-2) \\ \Rightarrow n^2+n &= 10n-20 \\ \Rightarrow n^2-9n+20 &= 0 \\ \Rightarrow (n-4)(n-5) &= 0 \\ \Rightarrow n &= 4 \text{ or } n = 5. \end{aligned}$$

6. Evaluate n if $22 \times {}^n P_5 = 7 \times {}^{n+2} P_5$

Solution : $22 \times {}^n P_5 = 7 \times {}^{n+2} P_5$

$$\begin{aligned} \Rightarrow 22 \times \frac{|n|}{|n-5|} &= 7 \times \frac{|n+2|}{|n+2-5|} \\ \Rightarrow 22 \times \frac{|n|}{|n-5|} &= 7 \times \frac{(n+2)(n+1)|n|}{|n-3|} \\ \Rightarrow \frac{22}{|n-5|} &= \frac{7(n+2)(n+1)}{(n-3)(n-4)|n-5|} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow 22(n-3)(n-4) = 7(n+2)(n+1) \\
&\Rightarrow 22n^2 - 154n + 264 = 7n^2 + 21n + 14 \\
&\Rightarrow 15n^2 - 175n + 250 = 0 \\
&\Rightarrow 3n^2 - 35n + 50 = 0 \\
&\Rightarrow (n-10)(3n-5) = 0 \\
&\Rightarrow n = 10 \text{ or } n = \frac{5}{3}
\end{aligned}$$

But n is a positive integer.

$$\therefore n = 10.$$

7. Prove that

$${}^{2n}P_n = 2^n \{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)\}$$

Solution :

$$\begin{aligned}
{}^{2n}P_n &= \frac{\underline{2n}}{\underline{2n-n}} \\
&= \frac{\underline{2n}}{\underline{n}} \\
&= \frac{2n(2n-1)(2n-2)\dots\dots\dots 4.3.2.1}{\underline{n}} \\
&= \frac{\{2n(2n-2)(2n-4)\dots\dots\dots 4.2\} \{(2n-1)(2n-3)\dots\dots 3.1\}}{\underline{n}} \\
&= \frac{2^n \{n(n-1)(n-2)\dots\dots\dots 2.1\} \{(2n-1)(2n-3)\dots\dots 3.1\}}{\underline{n}} \\
&= \frac{2^n \underline{n} \{1.3.5\dots\dots(2n-3)(2n-1)\}}{\underline{n}} \\
&= 2^n \{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)\}.
\end{aligned}$$

8. How many numbers between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5 without repetition? How many of them are divisible by 5?

Solution : Since the number lies between 100 and 1000 therefore it consists of 3 digits.

The digits in the unit place and the tenth place may be any one of the given 6 digits without repetition, but the digit 0 cannot be in the hundredth place.

Let us start with the hundredth place. The hundredth place can be filled up in 5 different ways (we can put any one from 1, 2, 3, 4, 5). After filling up the hundredth place, the tenth place can be filled up with the remaining 5 digits in 5 different ways. Now for the unit place there remain 4 digits, so the unit place can be filled up in 4 different ways.

Hence the required number of numbers

$$= 5 \times 5 \times 4 = 100$$

For the second part, the numbers divisible by 5 must end with 0 or 5.

Now 3 digit numbers ending with 0 can be obtained in 5P_2 ways (\because 0 is placed in the unit place, the other two places can be filled up with the remaining 5 digits in 5P_2 ways. Again 3 digit numbers ending with 5 but not beginning with 0 can be obtained in $4 \times 4 = 16$ ways. (since 5 is placed in the unit place and since 0 cannot be placed in the hundredth place, the hundredth place can be filled up in 4 ways and the tenth place can be filled up by any one of the remaining 4 digits in 4 ways)

\therefore Required number of numbers divisible by 5

$$= {}^5P_2 + 16 = 20 + 16 = 36.$$

Alternatively,

Since the number lies between 100 and 1000, it consists of 3 digits. With the given 6 digits, we can fill up three places in 6P_3 ways. But these include some numbers beginning with 0. Which are not 3 digit numbers. Total number of numbers beginning with zero is 5P_2 (since 0 is placed in the beginning, the remaining two places can be filled up with the remaining 5 digits in 5P_2 ways)

\therefore Required number of 3 digit numbers

$$= {}^6P_3 - {}^5P_2 = 120 - 20 = 100$$

Again numbers divisible by 5 must end with 0 or 5. Now, 3 digit numbers ending with 0 can be obtained in 5P_2 ways.

Also 3 digit numbers ending with 5 can be obtained in 5P_2 ways which include those numbers beginning with 0 also.

Now 3 digit numbers ending with 5 and beginning with 0 can be obtained in 4 ways (after filling up the 1st place by 0 and the last place by 5, the middle place can be filled up with the remaining 4 digits in 4 ways)

\therefore number of 3 digit numbers ending with 5

$$= {}^5P_2 - 4 = 20 - 4 = 16$$

\therefore Total number of 3 digit numbers divisible by

$$5 = {}^5P_2 + 16 = 20 + 16 = 36.$$

9. Suppose a license plate contains two distinct letters of the English alphabet followed by four digits, the first digit being non zero. How many different license plate can be obtained?

Solution : There are 26 different letters in English alphabet. The two different letters can be arranged in ${}^{26}P_2$ ways. Now the first digit may be any one of 1, 2, 3,, 9. So the first digit may be taken in 9 different ways. The 2nd, 3rd and the 4th digit may be any one of 0, 1, 2,, 9. Since the digits may be repeated, therefore each of 2nd, 3rd and 4th digits may be taken in 10 different ways.

$$\begin{aligned} \therefore \text{required number of plates} &= {}^{26}P_2 \times 9 \times 10 \times 10 \times 10 \\ &= \dots \times 9000 \\ &= 5850000 \end{aligned}$$

10. Find the number of ways in which the letters of the word COMMERCE can be arranged. So that the vowels may never be separated.

Solution : Consider the 3 vowels O, E, E together as one letter. Then we have total number of letters as 6 of which two are C's and 2 are M's.

\therefore number of arrangements of these 6 letters

$$= \frac{|6|}{|2| |2|}$$

But the 3 vowels O, E, E (of which two are same) can be arranged among themselves in

$$\frac{|3|}{|2|} \text{ ways.}$$

Hence required number of arrangements

$$\begin{aligned} &= \frac{|6|}{|2| |2|} \times \frac{|3|}{|2|} \\ &= 540. \end{aligned}$$

11. In how many ways can 10 answer scripts be arranged placing one on another so that the script getting the highest mark and the script getting the lowest mark donot come together?

Solution : First we shall find the number of arrangements in which the script getting the highest mark (say H) and the script getting the lowest mark (say L) come together.

Considering these two scripts (i.e. H and L) as one single script we have 9 scripts which

can be arranged in 9P_9 ways. Again H and L can be arranged among themselves in 2P_2 ways.

\therefore number of arrangements in which both H and L come together

$$= {}^9P_9 \times {}^2P_2 = 9 \times 2 = 2 \times 9$$

Again without any restriction all the 10 answer scripts can be arranged in ${}^{10}P_{10}$ ways.

Hence number of arrangements in which the particular two scripts do not come together

$$= {}^{10}P_{10} - 2 \times 9$$

$$= 10 - 2 \times 9$$

$$= 10 \times 9 - 2 \times 9$$

$$= 8 \times 9$$

$$= 2903040.$$

12. How many numbers not more than 5 digits can be formed with the digits 2, 3, 5?

Solution : The number may be of 1 digit or 2 digit or 3 digit or 4 digit or 5 digit. Now, 1 digit number can be formed in 3 ways (we may take any one from 2, 3 and 5)

\therefore number of 1 digit number = 3.

To construct a 2 digit number we are to fill up 2 places with three things where repetition is allowed. So each of the two places can be filled up in 3 ways.

\therefore number of 2 digit numbers = $3 \times 3 = 3^2$

Similarly number of 3 digit numbers = 3^3

number of 4 digit numbers = 3^4

number of 5 digit numbers = 3^5

Hence required number of numbers

$$= 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

EXERCISE

1. What do you mean by permutation?

2. Find the values of :

(i) 5P_3 , (ii) 6P_4 , (iii) 4P_2 , (iv) 5P_0 , (v) 6P_6 ,

Ans : (i) 60, (ii) 360, (iii) 12, (iv) 1, (v) 720.

3. Prove that

(i) ${}^nP_{r-1} = {}^{n-1}P_{r-1} + (r-1){}^{n-1}P_{r-2}$

(ii) ${}^{n-1}P_r = (n-r) \cdot {}^{n-1}P_{r-1}$

(iii) ${}^nP_r = (n-r+1) \cdot {}^nP_{r-1}$

(iv) ${}^{n+1}P_{r+1} = (n+1) \cdot {}^nP_r$

4. Find the value of n if

(i) ${}^nP_3 = 120$

(ii) ${}^{n+1}P_4 = 4 \times {}^nP_3$

(iii) ${}^nP_5 = 20 \times {}^nP_3$

(iv) ${}^nP_5 = 10 \times {}^{n-1}P_4$

Ans : (i) 6, (ii) 3, (iii) 8, (iv) 10

5. Find the value of r if

(i) ${}^{11}P_r = 110$

(ii) $7Pr = 840$

(iii) ${}^{50}P_{r+2} : {}^{50}P_{r-1} = 720 : 1$

Ans : (i) 2, (ii) 4, (iii) 41.

6. Find the value of n if

(i) ${}^nP_3 : {}^{n+2}P_3 = 5 : 12$

Ans : 7

(ii) ${}^{n+2}P_3 : {}^{n+1}P_2 = 5 : 1$

Ans : 3

(iii) ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$

Ans : 4

7. If ${}^{m+n}P_2 = 56$, ${}^{m-n}P_2 = 12$, find the values of m and n

Ans : m = 6, n = 2.

8. In how many ways can three friends be admitted in 5 selective colleges so that no two of them may read in the same college?

Ans : 60.

9. Two passengers enter into a bus where there are 5 vacant seats. In how many ways can they take the seats?

Ans : 20.

10. In how many ways can a man cast 4 votes among 5 candidates, when each candidate is entitled to receive all the 4 votes?
Ans : 625.
11. How many three digit numbers can be formed from the digits 0, 2, 3, 5, 6, 8 if no digit is repeated?
Ans : 100.
12. How many numbers less than 1000 can be formed with the digits 2, 3?
Ans : 14.
13. How many numbers less than 10000 can be formed with the digits 0, 2, 3?
Ans : 81.
14. How many numbers not more than 4 digits can be formed with the digits 3, 4?
Ans : 30.
15. How many numbers between 4000 and 5000 can be formed with the digits 0 to 9 if repetition is not allowed?
Ans : 504.
16. How many even numbers between 3000 and 4000 can be formed with the digits 0 to 9 if repetition is not allowed?
Ans : 280.
17. How many numbers greater than 20000 can be formed by using once each of the digits 1, 2, 3, 0, 2?
Ans : 36.
18. How many odd numbers of six digits can be formed with the digits 0 to 5 each digit occurring only once?
Ans : 288.
19. How many numbers between 1000 and 4000 can be formed with the digits 0, 1, 2, 3, 4 repetition of digits being allowed?
Ans : 375.
20. In how many ways can 4 different books be given to 5 boys if
(i) no boy may have more than one book?
(ii) one boy may have more than one book?
Ans : (i) 120, (ii) 625.
21. In how many ways can 8 examination scripts be arranged so that the best and the worst scripts always come together?
Ans : 10080.

22. In how many ways can 5 boys 3 girls be arranged so that no two girls sit together?

Ans : 14400.

23. In how many ways can 9 gentlemen and 9 ladies sit around a round table if no two gentlemen sit side by side?

Ans : $\frac{8 \times 9}{4}$

24. In how many ways can 15 doctors and 12 engineers be seated in a row so that no two engineers may occupy consecutive positions?

Ans : $= \frac{15 \times 16}{4}$

25. Find the number of ways in which 16 different books can be arranged on a shelf so that two particular books shall not be together.

Ans : 14×15

26. In how many ways can the letters of the word VOLUME be arranged so that the letters L and M can occupy only the even places?

Ans : 144.

27. Find the number of ways in which the letters of the word ENGLISH can be arranged so that the vowels occupy the odd places.

Ans : 1440.

28. In how many ways can the letters of the word 'EXAMINATION' be arranged?

Ans : $= \frac{11!}{2! 2! 2!} = 4989600$

29. In how many ways can the letters of the word 'EXAMINATION' be arranged taking all at a time so that EXM is always together and also in the given order?

Ans : 45360.

30. In how many ways can the letters of the word 'MATHEMATICS' be arranged so that the vowels may never be separated?

Ans : 120960.

31. Find the number of arrangements of the letters of the word 'BANANA' in which two N's donot appear adjacently.

Ans : 40.

32. There are three different books of Accountancy, 3 of Management, 2 of Mathematics and 2 of Statistics. In how many ways can these books be arranged in shelf so that books of the same subject are not separated?

Ans : 3456.

COMBINATION :

The different groups or collections or selections that can be made out of a given number of things by taking some or all of them at a time are called their combinations.

Suppose we are given three things A, B, C. Then the combinations of these 3 things taken one, two and three at a time are shown below :

one at a time	two at a time	three at a time
A	AB	ABC
B	AC	
C	BC	

It is to be noted that the combinations are independent of order. For example, AB and BA are two permutations while in case of combination they are same.

Notation : The number of combination of n different things taken r ($r \leq n$) at a time is denoted by ${}^n C_r$ or ${}_n C_r$ or C (n, r).

Thus we see that

$${}^3 C_1 = 3, {}^3 C_2 = 3 \text{ and } {}^3 C_3 = 1.$$

Now let us taken an example of 4 different things A, B, C, D. We see that combinations of these 4 things taken one, two, three and 4 at a time are as follows :

one at a time	2 at a time	3 at a time	4 at a time
A	AB	ABC	ABCD
B	AC	ABD	
C	AD	ACD	
D	BC	BCD	
	BD		

Thus we see that

$${}^4 C_1 = 4, {}^4 C_2 = 6, {}^4 C_3 = 4, {}^4 C_4 = 1$$

Theorem : The number of combinations of n different things taken r ($r \leq n$) at a time is

equal to $\frac{\underline{n}}{\underline{r} \underline{n-r}}$ i.e. ${}^n C_r = \frac{\underline{n}}{\underline{r} \underline{n-r}}$

Proof : Let the number of combinations of n different things taken r at a time be equal to x i.e. ${}^n C_r = x$.

Now, each of these x combinations contains r things. These r things can be arranged among themselves in ${}^r P_r = \underline{r}$ ways. Thus from each of the x combinations we may get \underline{r} permutations. So the number of permutations of n different things taken r at a time is $x \underline{r}$.

$$\therefore {}^n P_r = x \underline{r}.$$

$$\Rightarrow \frac{\underline{n}}{\underline{n-r}} = x \underline{r}$$

$$\Rightarrow x = \frac{\underline{n}}{\underline{r} \underline{n-r}}$$

Thus ${}^n C_r = \frac{\underline{n}}{\underline{r} \underline{n-r}}$

Corollary (1)

Putting $r = 0$ in ${}^n C_r = \frac{\underline{n}}{\underline{r} \underline{n-r}}$ we get

$${}^n C_0 = \frac{\underline{n}}{\underline{0} \underline{n-0}} = \frac{\underline{n}}{1 \cdot \underline{n}} = 1 \quad (\because \underline{0} = 1)$$

Corollary (2)

Putting $r = n$, we get

$${}^n C_n = \frac{\underline{n}}{\underline{n} \underline{n-n}} = \frac{\underline{1}}{\underline{0}} = 1$$

Thus ${}^n C_n = {}^n C_0 = 1$

Corollary (3)

Putting $n-r$ in place of r , we get

$${}^n C_{n-r} = \frac{\underline{n}}{\underline{n-r} \underline{n-(n-r)}} = \frac{\underline{n}}{\underline{n-r} \underline{r}} = {}^n C_r$$

Thus we get

${}^n C_{n-r} = {}^n C_r$

For example, ${}^7 C_4 = {}^7 C_3$, ${}^7 C_5 = {}^7 C_2$ etc.

We see that

$${}^n C_r = {}^n C_s \Rightarrow \text{either } r = s \text{ or } r+s = n$$

One important result

$${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$$

Proof :
$${}^n C_r + {}^n C_{r-1} = \frac{\underline{n}}{\underline{r} \underline{n-r}} + \frac{\underline{n}}{\underline{r-1} \underline{r-(r-1)}}$$

$$= \frac{\underline{n}}{r \underline{r-1} \underline{n-r}} + \frac{\underline{n}}{\underline{r-1} \underline{n-r+1}}$$

$$= \frac{\underline{n}}{r \underline{r-1} \underline{n-r}} + \frac{\underline{n}}{\underline{r-1} (n-r+1) \underline{n-r}}$$

$$= \frac{\underline{n}}{\underline{r-1} \underline{n-r}} \left[\frac{1}{r} + \frac{1}{n-r+1} \right]$$

$$= \frac{\underline{n}}{\underline{r-1} \underline{n-r}} \frac{n+1}{r(n-r+1)}$$

$$= \frac{\underline{n+1}}{\underline{r} \underline{(n+1)-r}}$$

$$= {}^{n+1} C_r$$

For Example,

$${}^8 C_5 + {}^8 C_4 = {}^9 C_5$$

$${}^9 C_6 + {}^9 C_5 = {}^{10} C_6$$

WORKED OUT EXAMPLES :

1. Find the value of

(i) ${}^n C_1$ (ii) ${}^n C_2$ (iii) ${}^n C_3$

Solution :

(i)
$${}^n C_1 = \frac{\underline{n}}{\underline{1} \underline{n-1}} = \frac{n \underline{n-1}}{\underline{n-1}} = n \quad (\because \underline{1} = 1)$$

$$(ii) {}^n C_2 = \frac{\underline{n}}{\underline{2} \underline{n-2}} = \frac{n(n-1)\underline{n-2}}{\underline{2} \underline{n-2}} = \frac{n(n-1)}{\underline{2}}$$

$$(iii) {}^n C_3 = \frac{\underline{n}}{\underline{3} \underline{n-3}} = \frac{n(n-1)(n-2)\underline{n-3}}{\underline{3} \underline{n-3}} = \frac{n(n-1)(n-2)}{\underline{3}}$$

Thus we see that

$${}^n C_1 = n, {}^n C_2 = \frac{n(n-1)}{\underline{2}}, {}^n C_3 = \frac{n(n-1)(n-2)}{\underline{3}}$$

Similarly we can see that

$${}^n C_4 = \frac{n(n-1)(n-2)(n-3)}{\underline{4}}, {}^n C_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{4}}$$

and so on.

2. Find the value of

$$(i) {}^9 C_4, \quad (ii) {}^8 C_5, \quad (iii) {}^7 C_0, \quad (iv) {}^{25} C_{20}$$

Solution :

$$(i) {}^9 C_4 = \frac{\underline{9}}{\underline{4} \underline{5}} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$$

$$(ii) {}^8 C_5 = \frac{\underline{8}}{\underline{5} \underline{3}} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

$$(iii) {}^7 C_0 = 1 (\because {}^n C_0 = 1)$$

$$(iv) {}^{25} C_{20} = \frac{\underline{25}}{\underline{20} \underline{5}} = \frac{25 \times 24 \times 23 \times 22 \times 21}{5 \times 4 \times 3 \times 2 \times 1} = 53130$$

3. Find the value of n if

$$(i) {}^n C_{n-2} = 6 \quad (ii) {}^n C_7 = {}^n C_5$$

Solution :

$$(i) {}^n C_{n-2} = 6$$

$$\Rightarrow \frac{\underline{n}}{\underline{n-2} \underline{n-(n-2)}} = 6$$

$$\Rightarrow \frac{n(n-1)}{2} = 6$$

$$\Rightarrow n(n-1) = 12 = 4 \times 3$$

$$\Rightarrow n = 4$$

(\because n is a positive integer)

$$(ii) \quad {}^n C_7 = {}^n C_5$$

$$\Rightarrow n = 7+5 = 12 \quad ({}^n C_r = {}^n C_s \Rightarrow r = s \text{ or } r+s = n)$$

4. If ${}^{24} C_{r+3} = {}^{24} C_{2r}$ find the value of r.

Solution :

$${}^{24} C_{r+3} = {}^{24} C_{2r}$$

$$\Rightarrow r+3 = 2r \quad \text{or } r+3+2r = 24$$

$$\Rightarrow r = 3 \quad \text{or } r = 7$$

\therefore Value of r is 3 or 7.

5. If ${}^n P_r = 72$ and ${}^n C_r = 36$, find n and r.

Solution : We have

$${}^n P_r = \underline{r} \times n C_r$$

$$\Rightarrow 72 = \underline{r} \times 36$$

$$\Rightarrow r = 2$$

$$\Rightarrow r = 2$$

Again ${}^n P_r = 72$

$$\Rightarrow {}^n P_2 = 72$$

$$\Rightarrow \frac{\underline{n}}{\underline{n-2}} = 72$$

$$\Rightarrow n(n-1) = 72 = 9 \times 8$$

$$\Rightarrow n = 9$$

\therefore n = 9 and r = 2

6. If ${}^n C_4 : {}^n C_7 = 7:2$, find the value of n.

Solution :

$$\frac{{}^n C_4}{{}^n C_7} = \frac{7}{2}$$

$$\Rightarrow \frac{\frac{n}{4(n-4)}}{\frac{n}{7(n-7)}} = \frac{7}{2}$$

$$\Rightarrow \frac{7(n-7)}{4(n-4)} = \frac{7}{2}$$

$$\Rightarrow \frac{7 \times 6 \times 5}{(n-4)(n-5)(n-6)} = \frac{7}{2}$$

$$\Rightarrow (n-4)(n-5)(n-6) = 60 = 5 \times 4 \times 3$$

(n-4, n-5, n-6 are three consecutive positive integers)

$$\Rightarrow n-4 = 5$$

$$\Rightarrow n = 9$$

7. If ${}^{n-1}C_3 : {}^nC_5 = 5:8$, find n.

Solution :

$$\frac{{}^{n-1}C_3}{{}^nC_5} = \frac{5}{8}$$

$$\Rightarrow \frac{\frac{n-1}{3(n-1-3)}}{\frac{n}{5(n-5)}} = \frac{5}{8}$$

$$\Rightarrow \frac{5(n-5)(n-1)}{3(n-4)2} = \frac{5}{8}$$

$$\Rightarrow \frac{5 \times 4}{(n-4)n} = \frac{5}{8}$$

$$\Rightarrow n(n-4) = 32$$

$$\Rightarrow n^2 - 4n - 32 = 32$$

$$\Rightarrow (n-8)(n+4) = 0$$

$$\Rightarrow n = 8 \text{ or } n = -4$$

But n is a positive integer. $\therefore n = 8$

8. Prove that

$${}^n C_r + {}^{n-1} C_{r-1} + {}^{n-1} C_{r-2} = {}^{n+1} C_r$$

Solution :

$$\begin{aligned} \text{L.H.S.} &= {}^n C_r + ({}^{n-1} C_{r-1} + {}^{n-1} C_{r-2}) \\ &= {}^n C_r + {}^n C_{r-1} \quad [\because {}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r] \\ &= {}^{n+1} C_r \\ &= \text{R.H.S.} \end{aligned}$$

9. A boy instead of writing ${}^n P_r$ writes ${}^n C_r$. By what number should he multiply his result to get the correct answer?

Solution : Since ${}^n P_r = {}^n C_r \times \underline{r}$, so he should multiply his result by \underline{r} to get the correct answer.

10. In an examination paper there are 10 questions and a candidate is to answer any 6 questions. In how many ways can a candidate select 6 questions?

Solution : The different way of selections of questions is nothing but number of combinations of 10 different things taken 6 at a time.

$$\therefore \text{required number of ways} = {}^{10} C_6$$

$$\begin{aligned} &= \frac{\underline{10}}{\underline{6} \underline{4}} \\ &= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\ &= 210. \end{aligned}$$

11. A question paper consists of 10 questions and a candidate is to answer 6 questions. If question no.1 is compulsory in how many ways can a candidate select his questions?

Solution : Since question no.1 is compulsory, therefore he is to select 5 questions out of remaining 9 questions.

$$\therefore \text{required number of ways} = \text{number of combinations of 9 things taken 5 at a time}$$

$$\begin{aligned} &= {}^9 C_5 \\ &= \frac{\underline{9}}{\underline{5} \underline{4}} \\ &= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \\ &= 126. \end{aligned}$$

12. Out of 5 teachers and 20 students in a school, a committee consisting of 3 teachers and 7 students is to be formed. In how many ways can this be done if

- (i) any teacher and any student may be included?
- (ii) one particular student cannot be in the committee.
- (iii) one particular teacher must be in the committee.

Solution :

(i) Since there is no restriction in selecting teachers and students, we can select 3 teachers out of 5 in 5C_3 ways and 7 students out of 20 in ${}^{20}C_7$ ways.

$$\begin{aligned} \therefore \text{required number of ways} &= {}^5C_3 \times {}^{20}C_7 \\ &= \frac{|5|}{|3| |2|} \times \frac{|20|}{|7| |13|} \\ &= \frac{5 \times 4}{2} \times \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14}{7 \times 6 \times 5 \times 4 \times 3 \times 2} \\ &= 775200. \end{aligned}$$

(ii) Since one particular student cannot be in the committee, so out of 19 students we are to select 7 students which can be done in ${}^{19}C_7$ ways.

$$\begin{aligned} \text{Hence the required number of ways} &= {}^5C_3 \times {}^{19}C_7 \\ &= \frac{|5|}{|3| |2|} \times \frac{|19|}{|7| |12|} \\ &= \frac{5 \times 4}{2} \times \frac{19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13}{7 \times 6 \times 5 \times 4 \times 3 \times 2} \\ &= 503880 \end{aligned}$$

(iii) Since one particular teacher must be in the committee, so we are to select 2 teachers out of remaining 4 teachers, which can be done in 4C_2 ways.

$$\begin{aligned} \text{Hence the required number of ways} &= {}^4C_2 \times {}^{20}C_7 \\ &= \frac{|4|}{|2| |2|} \times \frac{|20|}{|7| |13|} \\ &= 465120. \end{aligned}$$

13. There are 16 points in a plane, no three of which being in the same straight line. Find the number of straight line which can be drawn by joining these points. How many triangle can be formed by joining these points?

Solution : Since no three points are in a straight line, therefore by joining any two points out of 16, we can get a straight line.

$$\begin{aligned} \therefore \text{required number of straight lines} \\ &= {}^{16}C_2 \\ &= \frac{16 \times 15}{2} \\ &= 120 \end{aligned}$$

Again joining any three points we can form a triangle.

$$\begin{aligned} \therefore \text{required number of triangles} \\ &= {}^{16}C_3 \\ &= \frac{16 \times 15 \times 14}{3 \times 2 \times 1} \\ &= 560. \end{aligned}$$

14. There are 20 points in a plane, of which only 5 are collinear. How many (i) different straight lines, (ii) different triangles can be formed by joining these points?

Solution :

(i) By joining any two points we can get one straight line. Now, if out of 20 points no three points are collinear then we can get ${}^{20}C_2$ straight lines. But out of 20 points, 5 are collinear (i.e. lying in the same straight line). So from these 5 points we are getting only one straight line instead of getting 5C_2 straight lines.

$$\begin{aligned} \therefore \text{required number of straight lines} \\ &= {}^{20}C_2 - {}^5C_2 + 1 \\ &= \frac{20 \times 19}{2} - \frac{5 \times 4}{2} + 1 \\ &= 190 - 10 + 1 \\ &= 181. \end{aligned}$$

(ii) By joining any three non collinear points we can get a triangle. So if no three points out of the 20 points are collinear, then we can form ${}^{20}C_3$ number of triangles. But out of 20 points, 5 points are collinear. From these 5 points we cannot have any triangle, hence number of triangles lost = 5C_3 (if no three of these 5 points are collinear then number of triangles that can be formed by joining these 5 points = 5C_3)

∴ required number of triangles

$$\begin{aligned}
 &= {}^{20}C_3 - {}^5C_3 \\
 &= \frac{20 \times 19 \times 18}{3 \times 2 \times 1} - \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \\
 &= 1140 - 10 \\
 &= 1130.
 \end{aligned}$$

15. There are 20 points in a plane and A is one of them. If no three of these points are collinear, find the number of triangles that can be formed with A as one of the vertices.

Solution : Joining any three non collinear points we can form a triangle. But the point A must be a vertex of the triangle. So we are to select any two points out of the 19 points (excluding A from the given 20 points)

∴ required number of triangles

$$\begin{aligned}
 &= {}^{19}C_2 \\
 &= \frac{19 \times 18}{2} \\
 &= 171.
 \end{aligned}$$

16. Out of 9 engineers and 6 doctors how many different committees can be formed, each consisting of 6 engineers and 3 doctors?

Solution : 6 engineers can be selected out of 9 in 9C_6 ways. Again 3 doctors out of 6 can be selected in 6C_3 ways.

∴ required number of committees

$$\begin{aligned}
 &= {}^9C_6 \times {}^6C_3 \\
 &= \frac{|9}{|6| |3|} \times \frac{|6}{|3| |3|} \\
 &= \frac{|9}{|3| |3| |3|} \\
 &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 6} \\
 &= 1680.
 \end{aligned}$$

17. A committee consisting of 7 members is to be formed out of 8 engineers and 5 doctors. In how many ways can this be done if

(i) there are at least 3 engineers in a committee?

(ii) there are at least 2 engineers and one doctor in a committee?

Solution :

(i) Possible cases of formation of a committee are :

Engineers	Doctors	Number of selections
3	4	${}^8C_3 \times {}^5C_4 = 280$
4	3	${}^8C_4 \times {}^5C_3 = 700$
5	2	${}^8C_5 \times {}^5C_2 = 560$
6	1	${}^8C_6 \times {}^5C_1 = 140$
7	0	${}^8C_7 \times {}^5C_0 = 8$

$$\therefore \text{required number of ways} = 280+700+560+140+8 = 1688$$

(ii) Possible cases of formation of a committee are :

Engineers	Doctors	Number of selections
2	5	${}^8C_2 \times {}^5C_5 = 28$
3	4	${}^8C_3 \times {}^5C_4 = 280$
4	3	${}^8C_4 \times {}^5C_3 = 700$
5	2	${}^8C_5 \times {}^5C_2 = 560$
6	1	${}^8C_6 \times {}^5C_1 = 140$

$$\therefore \text{required number of ways} = 28+280+700+560+140 = 1708.$$

18. Mr. X has 7 friends consisting of 3 gentlemen and 4 ladies. His wife Mrs. X has also 7 friends consisting of 4 gentlemen and 3 ladies. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that Mr. X has 3 friends and Mrs. X has 3 friends.

Solution :

POSSIBLE CASES

Friends of Mr. X		Friends of Mrs. X		Number of Selections
Ladies	Gentlemen	Ladies	Gentlemen	
3	–	–	3	${}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$
2	1	1	2	${}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 = 324$
1	2	2	1	${}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 144$
–	3	3	–	${}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1$

$$\therefore \text{Total number of ways} = 16+324+144+1 = 485$$

EXERCISE

1. What do you mean by combination?

2. Find the values of :

(i) ${}^{10}C_6$ (ii) 9C_0 (iii) ${}^{24}C_{21}$ (iv) ${}^{10}C_6 + {}^9C_5 + {}^9C_4$

Ans : (i) 210, (ii) 1, (iii) 2024, (iv) 462

3. Find the value of r if

(i) ${}^{2n}C_r = {}^{2n}C_{r+2} + 2$

(ii) ${}^8C_r = {}^7C_r$

(iii) ${}^{25}C_{r+4} = {}^{25}C_{2r-3}$

Ans : (i) $n-1$, (ii) 0, (iii) 7 or 8

4. Find the value of n if

(i) ${}^nC_3 = {}^nC_5$

(ii) ${}^nC_{10} = {}^nC_{12}$

(iii) ${}^nC_3 : {}^{n-1}C_3 = 4:3$

(iv) ${}^nC_3 : {}^{n-1}C_4 = 8:5$

(v) ${}^{2n}C_3 : {}^nC_2 = 44:3$

(vi) ${}^nC_3 = 6 \times {}^nC_2$

(vii) ${}^{2n}C_3 : {}^nC_2 = 44:1$

Ans : (i) 8, (ii) 22, (iii) 12, (iv) 8, (v) 6, (vi) 20, (vii) 17

5. If ${}^nP_r = 336$ and ${}^nC_r = 56$, find n and r.

Ans : 8, 3.

6. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, find n and r.

Ans : 3, 2.

7. Prove that

(i) ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

(ii) ${}^{n-2}C_r + 2 \cdot {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2} = {}^nC_r$

8. A football team of 11 players is to be selected from a class of 40 students. In how many ways can the team be selected?

Ans : ${}^{40}C_{11}$

9. In how many ways can a person invite his 10 friends to a dinner by inviting any number of members?

Ans : 1023.

10. There are 12 questions in a question paper. In how many ways can a candidate select 8 questions?

Ans : 495.

11. There are 10 questions in a question paper. A candidate is to answer 6 questions but question no.1 and 10 are compulsory. In how many ways can a candidate select 6 questions in all?

Ans : 70.

12. A question paper contains 10 questions divided into two groups of 5 questions each. In how many ways can a candidate select 6 questions taking at least two questions from each group?

Ans : 200.

13. In an examination paper there are two groups each containing 4 questions. A candidate is required to answer 5 questions but not more than 3 questions from any group. Find the number of ways a candidate can select 5 questions in all.

Ans : 48.

14. A candidate is required to answer 6 questions out of 12 which are divided into two groups each containing 6 questions. In how many ways can he make his choice if he is not allowed to attempt more than 4 questions from any group?

Ans : 850.

15. A cricket team is to be formed from two groups A and B; group A consisting of 6 players and group B consisting of 8 players. In how many ways can the selection be made if at least 4 players from group A are included?

Ans : 344.

16. In a cricket club, there are 30 players of whom 15 are batsmen, 12 bowlers and 3 wicket keepers. In how many ways can a cricket team of 11 players be selected so as to include 6 batsmen, 4 bowlers and 1 wicket keeper?

Ans : 7432425.

17. A committee of 6 members is to be constituted from 6 gentlemen and 4 ladies. In how many ways can this be done so as to include at least one lady?

Ans : 209.

18. How many different triangles can be formed by joining the vertices of a polygon having 12 sides?

Ans : 220.

19. A polygon has 27 diagonals. Find the number of sides of the polygon.

Ans : 9.

20. Prove that the number of triangles which can be formed by joining the vertices of a polygon of n sides is $\frac{1}{6}n(n-1)(n-2)$. Also show that the polygon has $\frac{1}{2}n(n-3)$ diagonals.

Ans :

21. In how many ways can 22 players be divided into two teams each consisting of 11 players to play against each other?

Ans : $\frac{\underline{22}}{2\underline{11}^2}$

22. There are 15 points in a plane such that no three of them are collinear.

(i) How many line segments can be drawn by joining these points?

(ii) How many triangles will be formed by these line segments?

Ans : (i) 105, (ii) 455.

23. There are 18 points in a plane of which only 5 are collinear. How many (i) different straight lines, (ii) different triangles can be formed by joining them?

Ans : (i) 144, (ii) 806.

24. In how many ways can 6 books be divided equally among 3 students?

Ans : 90.

25. In how many ways can 10 books be divided equally between 2 students?

Ans : 252.

26. Every member of a group of friends gives New Years Greeting cards to each of his friends. If total number of cards they use is 132, how many members are there in the group?

Ans : 12.

27. Everybody in a party shakes hands with everybody else. If total number of handshakes is 120, find the number of persons in the party.

Ans : 16.

28. From a group of 7 gentlemen and 5 ladies a committee of 4 gentlemen and 3 ladies is to be formed. In how many ways can this be formed if one particular lady L_1 refuses to be in the committee if another particular L_2 be included?

Ans : 245.

PRINCIPLE OF MATHEMATICAL INDUCTION

Many statements or formulas involving natural numbers are to be established to be true for all natural numbers n . If we prove a statement $S(n)$ involving natural number n , to be true for $n=1, 2$ and 3 and conclude that $S(n)$ is true for all $n \in \mathbb{N}$, then our conclusion may not be true. For example, consider the statement.

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} + (n-1)(n-2)(n-3).$$

We see that the statement is true for $n=1, n=2$ and $n=3$ (putting $n=1, 2$ and 3 respectively we see that L.H.S.=R.H.S.). If we conclude that since the statement is true for $n=1, 2, 3$, hence it is true for all $n \in \mathbb{N}$, our conclusion is not true because if $n=4$, then

$$\text{L.H.S.} = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\text{R.H.S.} = \frac{4 \times 5 \times 9}{6} + 3 \times 2 \times 1 = 36.$$

So it is required to develop a method by which we can establish the truth of a given statement for all $n \in \mathbb{N}$. The principle lying behind the method is known as Principle of Mathematical Induction.

The Principle of Mathematical Induction may be stated as follows :

Given a mathematical statement or proposition $P(n)$ involving natural member n , if (i) $P(1)$ is true i.e. $P(n)$ is true for $n = 1$, and (ii) for any value k of n , we have

$$P(k) \text{ is true} \Rightarrow P(k+1) \text{ is true,}$$

then the proposition $P(n)$ will be true for all $n \in \mathbb{N}$.

EXPLANATION :

Let us suppose that we have proved (i) a proposition $P(n)$ to be true for $n=1$ and (ii) $P(n)$ to be true for $n=k+1$ if it is true for $n=k$.

Then by (i) $P(n)$ is true for $n=1$ and by (ii) since $P(n)$ is true for $n=1$, it is true for $n = 1+1 = 2$. Again since it is true for $n=2$, so it is true for $n = 2+1 = 3$ and so on. Continuing the process indefinitely, we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Note:

Let $P(n)$ be a statement. If we can prove that (i) $P(m)$ is true for a particular value m of $n \in \mathbb{N}$. and (ii) $P(k)$ is true $\Rightarrow P(k+1)$ is true for some arbitrary value $k \in \mathbb{N}$ and $k > m$; then $P(n)$ is true for all $n \geq m \in \mathbb{N}$.

WORKED OUT EXAMPLES :

1. Prove by the principle of mathematical induction that $1+2+3+\dots+n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$.

Solution :

When $n=1$, then L.H.S.=1 and R.H.S. = $\frac{1 \cdot 2}{2} = 1$

\therefore L.H.S. = R.H.S.

\therefore the proposition is true for $n = 1$.

Let the proposition be true for $n=k$ ($k > 1$)

$$\begin{aligned} \therefore 1 + 2 + 3 + \dots + k &= \frac{k(k+1)}{2} \\ = 1 + 2 + 3 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1) \left[\frac{k}{2} + 1 \right] \\ &= (k+1) \frac{k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

So the proposition is true for $n=k+1$ if it is true for $n=k$.

Hence by induction, the given statement is true for all $n \in \mathbb{N}$.

$$\therefore 1+2+3+ \dots + n = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$

2. If $n \in \mathbb{N}$, prove by mathematical induction that $a^n - b^n$ is divisible by $a - b$.

Solution : When $n=1$, then $a^n - b^n = a - b$ which is divisible by $a - b$.

So the proposition is true for $n=1$.

Let us suppose that the proposition is true for $n=k$. Then $a^k - b^k$ is divisible by $(a - b)$. So $a^k - b^k = c(a - b)$ where c is the quotient obtained on dividing $a^k - b^k$ by $(a - b)$.

Where c is the quotient obtained on dividing $a^k - b^k$ by $(a - b)$

$$\begin{aligned} \text{Now, } a^{k+1} - b^{k+1} &= a(a^k - b^k) + ab^k - b^{k+1} \\ &= ac(a - b) + b^k(a - b) \\ &= (a - b)(ac + b^k) \end{aligned}$$

Thus we see that $a^{k+1}-b^{k+1}$ is divisible by $(a-b)$. So the proposition is true for $n=k+1$ if it is true for $n=k$.

Hence by mathematical induction, the proposition is true for all $n \in \mathbb{N}$.

i.e. a^n-b^n is divisible by $(a-b)$ for all $n \in \mathbb{N}$.

3. Prove by mathematical induction that

$$\lfloor n \rfloor > 3^n \text{ for } n \geq 7.$$

Solution : When $n=7$, then $\lfloor n \rfloor = \lfloor 7 \rfloor = 5040$. and $3^n=3^7=2187$.

So the proposition is true for $n=7$.

Let the proposition be true for $n=k$ ($k > 7$).

Then $\lfloor k \rfloor > 3^k$ where $k > 7$

$$\Rightarrow (k+1) \lfloor k \rfloor > (k+1)3^k$$

$$\Rightarrow \lfloor k+1 \rfloor > 3 \cdot 3^k (\because k > 7 \therefore k+1 > 3)$$

$$\Rightarrow \lfloor k+1 \rfloor > 3^{k+1}$$

Thus the proposition is true for $n=k+1$ if it is true for $n=k$.

Hence by mathematical induction, the proposition is true for all $n \geq 7$.

i.e. $\lfloor n \rfloor > 3^n$ for $n \geq 7$.

EXERCISE

Prove by the principle of mathematical induction that–

1. Sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

2. $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$ for all $n \in \mathbb{N}$.

3. Sum of the first n odd natural numbers is n^2 .

4. Sum of the first n even natural numbers is $n(n+1)$.

5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$ for all $n \in \mathbb{N}$.

6. $1 + 4 + 7 + \dots + (3n-2) = \frac{1}{2} n(3n-1), \forall n \in \mathbb{N}$.

7. $1 + 2 + 4 + 8 + \dots + 2^{n-1} = 2^n - 1, \forall n \in \mathbb{N}.$

8. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}, n \in \mathbb{N}.$

9. $x^n - 1$ is divisible by $x - 1$ for all $n \in \mathbb{N}.$

10. $9^n + 7$ is divisible by 8 for all $n \in \mathbb{N}.$

11. $5^{2n} + 3n - 1$ is divisible by 9 for all $n \in \mathbb{N}.$

12. $2^n > n$ for all $n \in \mathbb{N}.$

13. $\lfloor n \rfloor > 2^n$ for $n \geq 4.$

14. $a^{2n} - b^{2n}$ is divisible by $a + b$ for all $n \in \mathbb{N}.$

15. $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{N}.$

BINOMIAL THEOREM

An expression involving two terms is called a **Binomial**. For example, $a+x$, $a+b$, $a + \frac{b}{x}$, $2x+3$, x^2+7x etc. are binomials or binomial expressions.

The theorem which gives us a general rule for expanding any power of a binomial expression is known as Binomial Theorem. We shall discuss here the Binomial theorem when the index i.e. power is a positive integer.

BINOMIAL THEOREM FOR A POSITIVE INTEGRAL INDEX :

When n is a positive integer, then for any real numbers a and x ,

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + {}^nC_3 a^{n-3}x^3 + \dots + {}^nC_r a^{n-r}x^r + \dots + {}^nC_n x^n \dots\dots\dots(i)$$

Proof : We shall prove this theorem with the help of principle of mathematical induction.

For $n = 1$,

L.H.S. of (i) = $(a+x)^1 = a+x$

R.H.S. of (i) = ${}^1C_0 a^1 + {}^1C_1 x^1 = a+x$

\therefore the proposition (i) is true for $n = 1$.

Let us suppose that the proposition be true for any arbitrary value m of n . Then

$$(a+x)^m = {}^mC_0 a^m + {}^mC_1 a^{m-1}x + {}^mC_2 a^{m-2}x^2 + {}^mC_3 a^{m-3}x^3 + \dots + {}^mC_r a^{m-r}x^r + \dots + {}^mC_m x^m$$

Multiplying both sides by $(a+x)$, we get

$$\begin{aligned} (a+x)^{m+1} &= (a+x) [{}^mC_0 a^m + {}^mC_1 a^{m-1}x + {}^mC_2 a^{m-2}x^2 + {}^mC_3 a^{m-3}x^3 + \dots \\ &\quad \dots + {}^mC_r a^{m-r}x^r + \dots + {}^mC_m x^m] \\ &= {}^mC_0 a^{m+1} + ({}^mC_0 + {}^mC_1) a^m x + ({}^mC_1 + {}^mC_2) a^{m-1} x^2 + ({}^mC_2 + {}^mC_3) a^{m-2} x^3 + \dots \\ &\quad \dots + ({}^mC_{r-1} + {}^mC_r) a^{m+1-r} x^r + \dots + {}^mC_m x^{m+1} \end{aligned}$$

But ${}^mC_0 = 1 = {}^{m+1}C_0$, ${}^mC_m = 1 = {}^{m+1}C_{m+1}$

and also ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

$\therefore {}^mC_0 + {}^mC_1 = {}^{m+1}C_1$

${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$ etc.

$$\begin{aligned} \therefore (a+x)^{m+1} = & {}^{m+1}C_0 a^{m+1} + {}^{m+1}C_1 a^m x + {}^{m+1}C_2 a^{m-1} x^2 \\ & + {}^{m+1}C_3 a^{m-2} x^3 + \dots + {}^{m+1}C_r a^{m+1-r} x^r + \dots + {}^{m+1}C_{m+1} x^{m+1} \end{aligned}$$

Thus we see that the proposition is true for $n = m+1$ if it is true for $n = m$.

Hence by mathematical induction the proposition is true for all $n \in \mathbb{N}$.

Hence if $n \in \mathbb{N}$, then

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + {}^nC_r a^{n-r} x^r + \dots + {}^nC_n x^n$$

SOME IMPORTANT POINT TO BE NOTED :

1. The number of terms in the expansion of $(a+x)^n$ is $(n+1)$ i.e. one more than the index n .
2. In each term of the expansion of $(a+x)^n$, the sum of the indices of a and x is equal to n .
3. The index of a in the first term is n and thereafter it goes on decreasing by one and it becomes zero in the last term. On the other hand, the index of x in the first term is zero and it goes on increasing by one and lastly it becomes n .

COEFFICIENTS OF EQUIDISTANT TERMS FROM BOTH ENDS

In the expansion of $(a+x)^n$, the binomial coefficients of terms equidistant from the beginning and the end are equal.

In the expansion of $(a+x)^n$ the $(r+1)^{\text{th}}$ term from the beginning is ${}^nC_r a^{n-r} x^r$ whose coefficient is nC_r . Again the $(r+1)^{\text{th}}$ term from the end is the $[(n+1)-r]^{\text{th}}$ term from the beginning i.e. $(n-r+1)^{\text{th}}$ term from the beginning. Therefore $(r+1)^{\text{th}}$ term from the end = $(n-r+1)^{\text{th}}$ term from the beginning.

$$= {}^nC_{n-r} a^{n-(n-r)} x^{n-r} = {}^nC_{n-r} a^r x^{n-r} \text{ whose coefficient is } {}^nC_{n-r}.$$

But we know that ${}^nC_r = {}^nC_{n-r}$

Hence the coefficients of the terms equidistant from the beginning and from the end are equal.

The Binomial Coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ in the expansion of $(a+x)^n$ follow a pattern for different values of n

$$\text{For } n = 0, \quad {}^0C_0 = 1$$

$$\text{For } n = 1, \quad {}^1C_0 = 1, {}^1C_1 = 1$$

$$\text{For } n = 2, \quad {}^2C_0 = 1, {}^2C_1 = 2, {}^2C_2 = 1$$

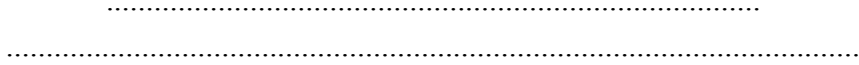
$$\text{For } n = 3, \quad {}^3C_0 = 1, {}^3C_1 = 3, {}^3C_2 = 3, {}^3C_3 = 1$$

$$\text{For } n = 4, \quad {}^4C_0 = 1, {}^4C_1 = 4, {}^4C_2 = 6, {}^4C_3 = 4, {}^4C_4 = 1$$

$$\text{For } n = 5, \quad {}^5C_0 = 1, {}^5C_1 = 5, {}^5C_2 = 10, {}^5C_3 = 10, {}^5C_4 = 5, {}^5C_5 = 1 \text{ and so on.}$$

We may arrange the coefficients in the form of a triangle called **Pascal's triangle**, as follows :

$$\begin{array}{cccccc}
 & & & & & & 1 \\
 & & & & & & & 1 & & 1 \\
 & & & & & & 1 & & 2 & & 1 \\
 & & & & & 1 & & 3 & & 3 & & 1 \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 \\
 1 & & 5 & & 10 & & 10 & & 5 & & 1
 \end{array}$$



The coefficients for different values of n are arranged serially in rows. it is observed that the numbers at the two ends in each row are 1 each and each intermediate number is the sum of the nearest two numbers in the row immediately above it.

This triangle gives us a simple rule for finding the coefficients of a binomial expansion, specially when n is not a big number.

The numbers in the 1st, 2nd, 3rd, rows give the binomial coefficients (in order) in the expansion of (a+x)ⁿ corresponding to n = 0, 1, 2, respectively.

Middle Term(s)

If n is even say n = 2m, then the total number of terms in the expansion of (a+x)ⁿ is 2m+1 which is odd. So the (m+1)th term is the middle term which is equal to

$${}^n C_m C_m a^{n-m} x^m \text{ i.e. } {}^n C_{\frac{n}{2}} a^{\frac{n-n}{2}} x^{\frac{n}{2}} = {}^n C_{\frac{n}{2}} a^{\frac{n}{2}} x^{\frac{n}{2}}$$

Again if n is odd say n = 2m+1, then total number of terms in (a+x)ⁿ will be 2m+2 which is even. So (m+1)th and (m+2)th terms are the two middle terms which are equal to

$$\begin{aligned}
 & {}^n C_m a^{n-m} x^m \quad \text{and} \quad {}^n C_{m+1} a^{n-m-1} x^{m+1} \\
 \text{i.e.} \quad & {}^n C_{\frac{n-1}{2}} a^{\frac{n+1}{2}} x^{\frac{n-1}{2}} \quad \text{and} \quad {}^n C_{\frac{n+1}{2}} a^{\frac{n-1}{2}} x^{\frac{n+1}{2}} \\
 \text{i.e.} \quad & {}^n C_{\frac{n-1}{2}} a^{\frac{n+1}{2}} x^{\frac{n-1}{2}} \quad \text{and} \quad {}^n C_{\frac{n+1}{2}} a^{\frac{n-1}{2}} x^{\frac{n+1}{2}}
 \end{aligned}$$

The General Term in the expansion of (a+x)ⁿ

In the expansion of (a+x)ⁿ if we denote the 1st term by t₁, 2nd term by t₂, 3rd term by t₃ etc. then

$$t_1 = {}^nC_0 a^n = {}^nC_0 a^{n-0} x^0$$

$$t_2 = {}^nC_1 a^{n-1} x = {}^nC_1 a^{n-1} x^1$$

$$t_3 = {}^nC_2 a^{n-2} x^2$$

.....

Thus $t_{r+1} = {}^nC_r a^{n-r} x^r$

We see that by putting $r=0, 1, 2, \dots, n$ in the $(r+1)^{th}$ term t_{r+1} , we get all the terms of the expansion.

$(r+1)^{th}$ term i.e. $t_{r+1} = {}^nC_r a^{n-r} x^r$ is called the **general term** in the expansion of $(a+x)^n$.

Deductions : In the binomial theorem

$$(a+x)^n = {}^nC_0 + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + {}^nC_3 a^{n-3} x^3 + \dots + {}^nC_r a^{n-r} x^r + \dots + {}^nC_n x^n$$

(i) if we put $(-x)$ place of x , we get

$$(a-x)^n = {}^nC_0 a^n - {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 - {}^nC_3 a^{n-3} x^3 + \dots + (-1)^r {}^nC_r a^{n-r} x^r + \dots + (-1)^n {}^nC_n x^n.$$

(ii) if we put $a = 1$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$$

(iii) if we put $a = 1$ and $(-x)$ in place of x , we get

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n.$$

WORKED OUT EXAMPLES :

1. By using Binomial theorem, expand $(2+x)^7$.

Solution :

$$\begin{aligned} (2+x)^7 &= {}^7C_0 2^7 + {}^7C_1 2^6 x + {}^7C_2 2^5 x^2 + {}^7C_3 2^4 x^3 + {}^7C_4 2^3 x^4 + {}^7C_5 2^2 x^5 + {}^7C_6 2 x^6 + {}^7C_7 x^7 \\ &= 2^7 + 7 \times 2^6 x + 21 \times 2^5 x^2 + 35 \times 2^4 x^3 + 35 \times 2^3 x^4 + 21 \times 2^2 x^5 + 7 \times 2 x^6 + x^7 \\ &= 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7 \end{aligned}$$

2. Evaluate 103^5 by using binomial theorem.

Solution :

$$\begin{aligned} 103^5 &= (100+3)^5 \\ &= 100^5 + {}^5C_1 100^4 \times 3 + {}^5C_2 100^3 \times 3^2 + {}^5C_3 100^2 \times 3^3 + {}^5C_4 100 \times 3^4 + 3^5 \\ &= 10^{10} + 5 \times 10^8 \times 3 + 10 \times 10^6 \times 9 + 10 \times 10^4 \times 27 + 5 \times 10^2 \times 81 + 243 \\ &= 10^{10} + 15 \times 10^8 + 90 \times 10^6 + 27 \times 10^5 + 405 \times 10^2 + 243 \\ &= 11, 59, 7, 40, 743. \end{aligned}$$

3. Expand $\left(2a - \frac{x}{2}\right)^7$.

Solution :

$$\begin{aligned} \left(2a - \frac{x}{2}\right)^7 &= (2a)^7 - {}^7C_1(2a)^6\left(\frac{x}{2}\right) + {}^7C_2(2a)^5\left(\frac{x}{2}\right)^2 - {}^7C_3(2a)^4\left(\frac{x}{2}\right)^3 \\ &\quad + {}^7C_4(2a)^3\left(\frac{x}{2}\right)^4 - {}^7C_5(2a)^2\left(\frac{x}{2}\right)^5 + {}^7C_6(2a)\left(\frac{x}{2}\right)^6 - {}^7C_7\left(\frac{x}{2}\right)^7 \\ &= 128a^7 - 7 \times 2^5a^6x + 21 \times 2^3a^5x^2 - 35 \times 2a^4x^3 + 35a^3 \frac{x^4}{2} \\ &\quad - 21a^2 \frac{x^5}{2^3} + 7a \frac{x^6}{2^5} - \frac{x^7}{2^7} \\ &= 128a^7 - 224a^6x + 168a^5x^2 - 70a^4x^3 + \frac{35}{2}a^3x^4 - \frac{21}{8}a^2x^5 + \frac{7}{32}ax^6 - \frac{x^7}{128}. \end{aligned}$$

4. Find the 8th term in the expansion of $\left(x^2 - \frac{2}{x}\right)^{10}$.

Solution : 8th term = t_{7+1}

$$\begin{aligned} &= {}^{10}C_7 (x^2)^{10-7} \left(-\frac{2}{x}\right)^7 \\ &= 120 x^6 \left(-\frac{128}{x^7}\right) \\ &= -\frac{15360}{x} \end{aligned}$$

5. Find the coefficient of x^5 in expansion of $\left(x - \frac{1}{x^2}\right)^{11}$.

Solution :

$$\begin{aligned} \text{Here } t_{r+1} &= {}^{11}C_r x^{11-r} \left(-\frac{1}{x^2}\right)^r \\ &= {}^{11}C_r x^{11-r} (-1)^r \frac{1}{x^{2r}} \\ &= (-1)^r {}^{11}C_r x^{11-3r} \end{aligned}$$

If this term contains x^5 then

$$\begin{aligned} 11-3r &= 5 \\ \Rightarrow r &= 2 \end{aligned}$$

\therefore term containing x^5 in t_{2+1}

$$\begin{aligned}\therefore \text{coefficient of } x^5 &= (-1)^2 {}^{11}C_2 \\ &= \frac{11 \times 10}{2} \\ &= 55.\end{aligned}$$

6. Find the coefficient of x^{16} in the expansion of $(x-x^2)^{10}$.

$$\begin{aligned}\text{Solution : Here } t_{r+1} &= {}^{10}C_r x^{10-r}(-x^2)^r \\ &= (-1)^r {}^{10}C_r x^{10-r+2r} \\ &= (-1)^r {}^{10}C_r x^{10+r}\end{aligned}$$

This term will contain x^{16} if $10+r = 16$ i.e. $r = 6$

$$\begin{aligned}\text{coefficient of } x^{16} &= (-1)^6 {}^{10}C_6 \\ &= \frac{10!}{6!4!} \\ &= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} \\ &= 210.\end{aligned}$$

7. Evaluate $(3+\sqrt{5})^5 + (3-\sqrt{5})^5$

$$\text{Solution : } (3+\sqrt{5})^5 = 3^5 + {}^5C_1 3^4 \sqrt{5} + {}^5C_2 3^3 \sqrt{5}^2 + {}^5C_3 3^2 \sqrt{5}^3 + {}^5C_4 3 \sqrt{5}^4 + \sqrt{5}^5.$$

$$\text{and } (3-\sqrt{5})^5 = 3^5 - {}^5C_1 3^4 \sqrt{5} + {}^5C_2 3^3 \sqrt{5}^2 - {}^5C_3 3^2 \sqrt{5}^3 + {}^5C_4 3 \sqrt{5}^4 - \sqrt{5}^5$$

$$\begin{aligned}\therefore (3+\sqrt{5})^5 + (3-\sqrt{5})^5 &= 2[3^5 + {}^5C_2 3^3 \sqrt{5}^2 + {}^5C_4 3 \sqrt{5}^4] \\ &= 2[243 + 10 \times 27 \times 5 + 5 \times 3 \times 25] \\ &= 2[243 + 1350 + 375] \\ &= 3936.\end{aligned}$$

8. Find the term containing x^4 in $\left(2x^2 + \frac{1}{3x}\right)^8$

Solution :

$$\text{Here } t_{r+1} = {}^8C_r (2x^2)^{8-r} \left(\frac{1}{3x}\right)^r$$

$$= {}^8C_r 2^{8-r} x^{16-2r} \frac{1}{3^r x^r}$$

$$= {}^8C_r \frac{2^{8-r}}{3^r} x^{16-3r}$$

This term will contain x^4 if $16-3r = 4$ i.e. $r = 4$

$$\therefore \text{the term containing } x^4 = t_{4+1}$$

$$= {}^8C_4 x^4$$

$$= {}^8C_4 \frac{2^{8-4}}{3^4} x^4$$

$$= 70 \times \frac{16}{81} x^4$$

$$= \frac{1120}{81} x^4.$$

9. Find the term independent of x in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^{10}$.

Solution : Here $t_{r+1} = {}^{10}C_r \left(\frac{x^3}{2}\right)^{10-r} \left(-\frac{2}{x^2}\right)^r$

$$= {}^{10}C_r \frac{x^{30-3r}}{2^{10-r}} (-1)^r \frac{2^r}{x^{2r}}$$

$$= (-1)^r {}^{10}C_r 2^{r-10+r} x^{30-5r}$$

This term will be independent of x if $30-5r = 0$ i.e. $r = 6$.

$$\therefore \text{term independent of } x = (-1)^6 {}^{10}C_6 2^{6-10+6}$$

$$= {}^{10}C_6 \times 2^2$$

$$= 210 \times 4$$

$$= 840.$$

10. Find the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{10}$.

Solution : The expansion contains $10+1 = 11$ terms.

\therefore the middle term is the 6th term.

$$\therefore \text{middle term} = t_6$$

$$= t_{5+1}$$

$$= {}^{10}C_5 x^{10-5} \left(-\frac{1}{x}\right)^5$$

$$= -{}^{10}C_5$$

11. Find the 4th term from the end in the expansion of $\left(\frac{x^3}{2} + \frac{2}{x^2}\right)^9$.

Solution : There are $9+1=10$ terms in the expansion. Therefore 4th term from the end is the 7th term from the beginning.

\therefore the required term $= t_7 = t_{6+1}$

$$= {}^9C_6 \left(\frac{x^3}{2}\right)^{9-6} \left(\frac{2}{x^2}\right)^6$$

$$= {}^9C_6 \frac{x^9}{2^3} \cdot \frac{2^6}{x^{12}}$$

$$= {}^9C_6 2^3 \frac{1}{x^3}$$

$$= \frac{672}{x^3}.$$

12. In the binomial expansion $(a+b)^{10}$ the coefficient of the $(4r+5)^{\text{th}}$ term is equal to the coefficient of the $(2r+1)^{\text{th}}$ term; find r .

Solution : $t_{4r+5} = {}^{10}C_{4r+4} a^{10-4r-4} b^{4r+4}$

and $t_{2r+1} = {}^{10}C_{2r} a^{10-2r} b^{2r}$

According to question,

$${}^{10}C_{4r+4} = {}^{10}C_{2r}$$

$$\Rightarrow 4r+4 = 2r \text{ or } 4r+4+2r = 10$$

$$\Rightarrow r = -2 \quad \text{or } r = 1$$

But $r = -2$ is not admissible, therefore $r = 1$. [$r = -2 \Rightarrow 4r+5 = -3$]

EXERCISE

1. Write down the number of terms in the expansion of

(i) $(3x+5y)^{11}$ (ii) $\left(2x - \frac{2}{x}\right)^7$
 (iii) $(1+2x+x^2)^8$ (iv) $\left(\frac{x}{a} + \frac{a}{x}\right)^{25}$

Ans : (i) 12, (ii) 8, (iii) 17, (iv) 26.

2. Using Binomial theorem expand the following :

(i) $(1+x)^6$ (ii) $\left(2x - \frac{1}{x^5}\right)^7$
 (iii) $\left(x^2 + \frac{1}{x^2}\right)^8$ (iv) $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$
 (v) $(x+y)^{10} + (x-y)^{10}$ (vi) $(2x+3y)^5$

Ans :

(i) $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
 (ii) $128x^7 - 448x + \frac{672}{x^5} - \frac{560}{x^{11}} + \frac{280}{x^{17}} - \frac{84}{x^{23}} + \frac{14}{x^{29}} - \frac{1}{x^{35}}$
 (iii) $x^{16} + 8x^{12} + 28x^8 + 56x^4 + 70 + \frac{56}{x^4} + \frac{28}{x^8} + \frac{8}{x^{12}} + \frac{1}{x^{16}}$
 (iv) $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$
 (v) $2x^{10} + 90x^8y^2 + 420x^6y^4 + 420x^4y^6 + 90x^2y^8 + 2y^{10}$
 (vi) $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

3. By using Binomial theorem, find the value of

(i) 9^6 (ii) 11^5 (iii) 101^7 (iv) 103^4 (v) 98^6 (vi) 99^5

Ans : (i) 531441 (ii) 161051 (iii) 107213535210701

(iv) 112550881 (v) 885842380864 (vi) 9509900499

4. Find the 6th term in the expansion of $\left(2x + \frac{3}{x}\right)^{12}$.

Ans : $24634368x^2$

5. Find the 8th term in $\left(1 + \frac{1}{x}\right)^{17}$.

Ans : ${}^{17}C_7 \frac{1}{x^7}$

6. Find the 6th term in the expansion of $\left(1 - \frac{3}{x^2}\right)^{10}$.

Ans : $-{}^{10}C_5 \times \frac{243}{x^{10}}$

7. Find the r^{th} term in expansion of $\left(x - \frac{1}{x}\right)^n$.

Ans : $(-1)^{r-1} {}^nC_{r-1} x^{n-2r+2}$

8. Find the n^{th} term in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$.

Ans : $\frac{{}^{2n}C_{n-1}}{n-1} x^2$

9. Find the coefficient of x^{15} in the expansion of $(2x - x^2)^{10}$.

Ans : -8064

10. Find the coefficient of $\frac{1}{x}$ in the expansion of $\left(x - \frac{1}{x^2}\right)^8$.

Ans : -56

11. Find the coefficient of x^2 in the expansion of $\left(2x + \frac{1}{x^2}\right)^8$.

Ans : 1792

12. Find the term containing x^4 in the expansion of $\left(x + \frac{a}{x}\right)^{10}$.

Ans : $120a^3x^4$

13. Find the term containing x^{18} in the expansion of $(x^2-3x)^{12}$.

Ans : $673596x^{18}$

14. Find the term containing a^{-10} in the expansion of $\left(a^2 - \frac{1}{a^2}\right)^{13}$.

Ans : $-715a^{-10}$

15. Find the term containing y^7 in $(x-2y)^{15}$.

Ans : $\frac{|5}{|7|8} 2^7x^8y^7$

16. Find the middle term(s) in the expansion of

(i) $\left(x + \frac{1}{x}\right)^{12}$ (ii) $\left(x - \frac{1}{x}\right)^9$ (iii) $\left(\frac{x}{a} + \frac{a}{x}\right)^{10}$

(iv) $(a+x)^{21}$ (v) $\left(x^3 + \frac{1}{x^2}\right)^7$ (vi) $(2y+3x)^8$

Ans : (i) 924, (ii) $126x, \frac{-126}{x}$, (iii) 252,
 (iv) $352716 a^{11}x^{10}, 352716 a^{10}x^{11}$
 (v) $35x^6, 35x$, (vi) $90720 x^4y^4$

17. Find the term independent of x in the expansion of

(i) $\left(x^2 - \frac{1}{x}\right)^{12}$, (ii) $\left(3x^2 + \frac{1}{3x}\right)^9$,

(iii) $\left(\sqrt{\frac{x}{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$, (iv) $\left(x - \frac{1}{x}\right)^{2m}$,

(v) $\left(\frac{4x^2}{3} - \frac{3}{2x}\right)^9$

Ans : (i) 495, (ii) $\frac{28}{9}$, (iii) $\frac{5}{12}$, (iv) $(-1)^m {}^{2m}C_m$, (v) 2268

18. Find the 5th term from the end in the expansion of $\left(x - \frac{1}{x}\right)^{12}$.

Ans : $\frac{495}{x^4}$

19. In the binomial expansion of $(a+x)^n$, the coefficient of 4th and 13th terms are equal to each other, find n.

Ans : 15.

20. Find the value of k if the constant term in the expansion of $\left(\sqrt{x} + \frac{k}{x^2}\right)^{10}$ is 405.

Ans : ± 3 .

21. In the expansion of $(x+1)^{41}$ the coefficient of $(2r+2)^{\text{th}}$ term is equal to the coefficient of $(4r-1)^{\text{th}}$ term; find r.

Ans : 7.

22. If the 21st and the 22nd terms in the expansion of $(1-x)^{44}$ are equal; find x.

Ans : $-\frac{7}{8}$.

23. In the expansion of $\left(a + \frac{x}{3}\right)^9$, the coefficient of x^2 and x^3 are equal; find a.

Ans : $\frac{7}{9}$.

24. Find the coefficient of x^6y^5 in the expansion of $(x+y)^{11}$.

Ans : 462.

25. In the expansion of $(1+x)^{m+n}$ where $m, n \in \mathbb{N}$, prove that the coefficients of x^m and x^n are equal.

26. Find the last three terms in the expansion of $(x+x^2)^8$.

Ans : $28x^{14}$, $8x^{15}$, x^{16} .

UNIT-3**3.1 SET THEORY****3.1.1. INTRODUCTION :**

In the later part of 19th century, George Cantor, a German mathematician introduced 'Set Theory' as a new discipline of Mathematics. Now a days, Set Theory is one of the basic theories of Mathematics which is used in every branch of knowledge like science, social-science, commerce etc. where use of Mathematics has become essential.

3.1.2. CONCEPT OF SET AND ELEMENTS :

According to Cantor's concept, A set is a collection of well-defined and distinct objects.

Here we assume that the word set is synonymous with the words 'collection', 'aggregate', 'class' and is comprised of elements. The words 'element', 'object', 'member' are synonymous.

In the definition, given above, the word well-defined means that given any object it must be possible to tell beyond doubt whether that object belongs to the collection or not. By distinct we mean that no two elements of the set are same e.g.

- (i) The collection of all odd numbers from 1 to 20 is a set. Because one can immediately say that 19 is a member of that collection but 16 is not a member.
- (ii) The collection of all good cricket players of India is not a set since the term 'good player' is vague and it is not well defined.

Similarly the collection of all beautiful girls, studying in Cotton College is not a set. However the collection of all girls studying in Cotton College is a set.

- ★ The objects which are listed in a set are called the elements of the set.
- ★ A set is named by using capital letter of English alphabet.
e.g. $G = \text{Set of students of Gauhati Commerce College.}$
- ★ The elements of a set are usually denoted by small letters of English alphabet.
- ★ To indicate that a particular element is a member of a set, we use the Greek symbol Capital Epsilon (\in) e.g. if an element 'a' is a member of the set 's', we write it as $a \in s$ (read as 'a belongs to s')

On the other hand, if we want to express that a particular element is not a member of that set, we use the symbol \notin e.g. if b is not a member of the set 's', we express it as $b \notin s$ (read as 'b does not belong to s')

Some examples of set :

1. The collection of vowels of English alphabet. This set contains five elements namely a, e, i, o, u.
2. The collection of first five prime numbers is a set containing the elements 2, 3, 5, 7, 11.
3. The collection of all districts of Assam is a set.
4. The collection of all past prime ministers of Indian Union is a set.

3.1.3. REPRESENTATION OF A SET :

A set is represented or described by using any one of the following two methods–

(i) **Roaster or Tabular Method :** In this method, the elements of a set are listed in any order with braces {}, being separated by commas, e.g. if V is the set of vowels of English alphabet, then it is written as

$$V = \{a, e, i, o, u\} \text{ or}$$

$$V = \{e, u, i, a, o\}$$

(ii) **Rule or Set Builder Method :** In this method a set is described by a characterizing property $P(x)$ of its elements x . In such a case, the set is described by $\{x \mid P(x) \text{ holds}\}$ or $\{x : P(x) \text{ holds}\}$ which is read as ‘the set of all x such that $P(x)$ holds’. The symbol ‘:’ or ‘|’ is read as ‘such that’. e.g. $V = \{x : x \text{ is a vowel of English alphabet}\}$

- Note :** (i) Repetition of the elements in a set is not allowed as by definition the elements of a set must be distinct. e.g. $P = \{1, 2, 3, 5, 5, 5\}$ is same as $P = \{1, 2, 3, 5\}$
- (ii) It should be kept in mind that out of the twenty six English alphabets, the alphabets C, I, N, Q, R, W, Z are reserved for the sets mentioned below

N = Set of Natural numbers

W = Set of Whole numbers

I or Z = Set of Integers

Q = Set of Rational numbers

R = Set of Real numbers

C = Set of Complex numbers.

[we have discussed about these numbers in the chapter ‘Real Number System’ in H.S. 1st year class]

WORKED OUT PROBLEMS :

Example 1 : Write the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}\right\}$ in the set builder method.

Solution : We observe that each element in the given set has the denominator one more than that of the numerator. Also the numerator begins from 1 and do not exceed 9. Hence in the st builder method the given set may be written as

$$\left\{x : x = \frac{n}{n+1}, n \in \mathbb{N}, n \leq 9\right\}$$

Example 2 : Write the set $A = \{x : x \in \mathbb{Z}, x^2 < 25\}$ in the roaster form.

Solution : We observe that the squares of the integces $0, \pm 1, \pm 2, \pm 3, \pm 4$ are less than 25. Therefore, the set A in roaster form is

$$A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

3.1.4. TYPES OF SETS :

Singleton Set : A set having only one element is called a singleton Set e.g.

- (i) The set of Lady Prime Minister of India = {Indira Gandhi}
- (ii) The set of natural satellite of the earth = {moon}
- (iii) The set of even prime number = {2}

All the above three sets are singleton set.

Finite Set : A set is called a finite set if its elements can be listed (counted or labelled) by natural numbers 1, 2, 3, and the process terminates at certain natural number (say n)

Cardinal number of a finite Set : The number n in the above definition is called the cardinal number of a finite set A and is denoted by $n(A)$

e.g. The set of days in a week is a finite set. If we denote the set by D, then $n(D) = 7$.

Infinite Set : A set whose elements can not counted by using any natural number is called an infinite set.

e.g. Set of all points in a plane is an infinite set.

3.1.4. EQUIVALENT SETS :

Two finite sets A and B are said to be equivalent if $n(A) = n(B)$ i.e. they contain the same number of elements and is denoted by $A \sim B$.

Eg. If $A = \{a, b, c\}$

$B = \{1, 2, 3\}$

Here $n(A) = n(B) = 3$

3.1.5. EQUAL SETS :

Two sets A and B are said to be equal if every element of A is a member of B and every element of B is a member of A and is denoted by $A=B$.

E.g. If $A = \{1, 2, 5, 6\}$
 $B = \{5, 2, 6, 1\}$

Then $A=B$

when A and B are not equal we write $A \neq B$.

Note : If $A=B$ then $A \sim B$.

But $A \sim B$ does not always imply that $A=B$.

Example 3 : Which of the following sets are finite and which are infinite :

- (i) $A = \{x : x \in \mathbb{Z} \text{ and } x^2 - 7x + 12 = 0\}$
- (ii) $B = \{x : x \in \mathbb{Z} \text{ and } x^2 \text{ is odd}\}$
- (iii) $D = \{x : x \in \mathbb{Z} \text{ and } x^2 = 64\}$
- (iv) $E = \{x : x \in \mathbb{Z} \text{ and } x < 3\}$

Solution : We have

(i) $A = \{x : x \in \mathbb{Z} \text{ and } x^2 - 7x + 12 = 0\} = \{3, 4\}$ So, A is a finite set.

(ii) $B = \{x : x \in \mathbb{Z} \text{ and } x^2 \text{ is odd}\}$.

$= \{\dots\dots\dots -7, -5, -3, -1, 1, 3, 5, 7 \dots\dots\dots\}$

clearly B is an infinite set

(iii) $D = \{x : x \in \mathbb{Z} \text{ and } x^2 = 64\} = \{-8, 8\}$

clearly D is a finite set.

(iv) $E = \{x : x \in \mathbb{Z} \text{ and } x < 3\}$

$= \{\dots\dots\dots -3, -2, -1, 0, 1, 2\}$

clearly E is an infinite set.

Example 4 : Which of the following pairs of sets are equal and which are equivalent?

$A = \{x : x \text{ is a letter of the word 'FLOW'}\}$

$B = \{x : x \text{ is a letter of the word 'FOLLOW'}\}$

$L = \{x : x \text{ is a letter of the word 'LATER'}\}$

$M = \{x : x \text{ is a letter of the word 'CIRCLE'}\}$

Solution :

Here $A : \{f, l, o, w\}$

$B = \{f, o, l, w\}$

$$L = \{l, a, t, e, r\}$$

$$M = \{c, i, r, l, e\}$$

So, $A = B$ and $L \sim M$

3.1.6 OPERATION ON SETS :

Like addition, multiplication, subtraction and other operations on numbers in arithmetic, there are certain operations on sets— mainly union, intersection and difference by using which we may combine any two sets.

3.1.6.1 UNION OF TWO SETS : Let A and B are any two sets. The union of A and B is the set consisting of all those elements, each one of which is contained either in A or in B or in both A and B and is denoted by $A \cup B$ (read as A union B)

$$\text{Thus } A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$\text{Consequently, } x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$$

$$\text{and } x \notin A \cup B \Rightarrow x \notin A \text{ or } x \notin B$$

$$\text{Imp : (i) } A \cup B = B \cup A$$

$$\text{(ii) } A \cup A = A$$

$$\text{E.g. If } A = \{a, b, c, d, e\}$$

$$B = \{b, e, f, g\}$$

$$\text{Then } A \cup B = \{a, b, c, d, e, f, g\}$$

3.1.6.2 INTERSECTION OF TWO SETS : Let A and B are any two sets. The intersection of A and B is the set consisting of all those elements which belongs to both A and B and is denoted by $A \cap B$ (read as intersection B).

$$\text{Thus } A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$$\text{Consequently, } x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

$$\text{and } x \notin A \cap B \Rightarrow x \notin A \text{ or } x \notin B$$

$$\text{E.g. If } A = \{a, b, c, d, e\}$$

$$B = \{b, e, f, g\}$$

$$\text{Then } A \cap B = \{b, e\}$$

$$\text{Imp : (i) } A \cap B = B \cap A$$

$$\text{(ii) } A \cap A = A$$

3.1.6.3 DIFFERENCE OF TWO SETS : Let A and B are any two sets. The difference of A and B is the set consisting of all those elements which belongs to A but does not belong to B and is denoted by $A-B$ or A/B (read as A difference B)

Similarly, the difference of B and A is the set consisting of all those elements which belong to B but does not belong to A and is denoted by $B-A$ or B/A (read as B difference A).

$$\text{Thus } A-B = \{x \mid x \in A \text{ but } x \notin B\}$$

$$\text{and } B-A = \{x \mid x \in B \text{ but } x \notin A\}$$

$$\text{E.g. If } A = \{a, b, c, d, e\}$$

$$\text{and } B = \{b, e, f, g\}$$

$$\text{Then } A-B = \{a, c, d\}$$

$$\text{and } B-A = \{f, g\}$$

Note : To find the difference of any two sets A and B, first mark the elements common to both A and B. Then to get $A-B$ write the remaining elements of A. Similarly, to get $B-A$ write the remaining elements of B.

$$\text{Imp: } A-B \neq B-A$$

3.1.6.4 SYMMETRIC DIFFERENCE OF TWO SETS : Let A and B are any two sets. The symmetric difference of A and B is denoted by $A\Delta B$ (read as A symmetric difference B) where $A\Delta B = (A-B) \cup (B-A)$.

$$\text{E.g. If } A = \{a, b, c, d, e\}$$

$$B = \{b, e, f, g\}$$

$$\text{Then, } A\Delta B = (A-B) \cup (B-A)$$

$$= \{a, c, d\} \cup \{f, g\}$$

$$= \{a, c, d, f, g\}$$

3.1.7 CONCEPT OF NULL OR EMPTY OR VOID SET :

From the above discussion, it is clear that intersection of any two sets is also a set.

Let us suppose that

$$A = \{2, 6, 9\}$$

$$B = \{4, 5\}$$

By definition of intersection, $A \cap B$ must be a set. But in this case, there is no element

common to both A and B. Hence we may come across a situation that there may be a set having no element which contradicts the very defⁿ of set that a set is a collection of objects.

So, the mathematicians have categorised this type set as a special type of set and it is named as Null Set.

Thus, a set having no element is called a Null Set, Void Set or Empty Set. A null set is generally denoted by the Greek alphabet ϕ (pronounced phy)

In Roaster Method it is denoted by $\{ \}$

- E.g. (i) The set of odd numbers lying between 9 and 11.
 (ii) The set of integers lying between 5 and 6.
 (iii) The set of even prime numbers > 2 .

Disjoint Set : Any two sets A and B are said to be disjoint if they have no common element i.e. $A \cap B = \phi$.

E.g. Set of positive integers and set of negative integers are disjoint sets as there is no integer which is both positive as well as negative

3.1.8 SUBSETS AND SUPERSETS :

Let A and B are any two sets such that every element of A is in B, then we say that A is a Subset of B and we write it as $A \subseteq B$ (read as A is a subset of B)

Also if A is a subset B then B is said to be a superset of A and denoted by $B \supseteq A$)

E.g. If $A = \{1, 3, 4, 7\}$

$B = \{1, 2, 3, 4, 5, 6, 7\}$

Then $A \subseteq B$ and $B \supseteq A$

Note : (i) If there exists a single element in A which is not in B, then A is not a subset of B and we express it as $A \not\subseteq B$.

(ii) Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$ conversely, if $A = B$, then $A \subseteq B$ & $B \subseteq A$.

E.g. $A = \{1, 2, 5, 6\}$

$B = \{1, 2, 3, 4, 5\}$

Here $A \not\subseteq B$ as $6 \in A$ but $6 \notin B$.

3.1.9 PROPER SUBSET :

If A and B are any two sets that every element of A is in B and B contains at least one element which is not in A, then A is said to be a proper subset of B and we write it as $A \subset B$.

E.g. $N \subset I$

where N = Set of natural nos

I = Set of Integer

- Note :** (i) If a subset is not proper it is said to be an improper subset.
 (ii) Every set is a subset of itself but it is not proper subset.
 (iii) Null set ϕ is a proper subset of every set.
 (iv) Every non-empty set A has at least two subsets – A and ϕ of which A is improper and ϕ is proper.
 (v) ϕ has no proper subset.

3.1.10 NUMBER OF SUBSETS : If a set contain n elements, it has got 2^n subsets.

3.1.11 FAMILY OF SET OR SET OF SETS OR CLASS OF SETS :

If all the elements of a set are sets themselves, then it is called a set of sets, family of set or class of sets.

E.g. Set of all lines in a plane is a family of set since each line itself is a set of points.

3.1.12 POWER SET :

The family or collection of all subsets of a given set A is called the power set of A and is denoted by $P(A)$

E.g. If $A = \{1, 2, 3\}$

Then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$,

Imp : If $n(A) = k$, $n[P(A)] = 2^k$

3.1.13 UNIVERSAL SET :

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a superset of each of the given sets. Such a set is called the Universal set and is denoted by U .

E.g. If $A = \{1, 2, 3\}$

$B = \{2, 4, 5, 6\}$

$C = \{1, 3, 6, 8, 9\}$

Then $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ may be taken as the universal set.

Similarly, $U_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$U_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

..... are also may be taken as the universal set.

Note : Thus universal set is not unique. There may be infinite number of universal sets.

3.1.14. COMPLEMENT OF A SET :

Let A be any set and U be its universal set. Then the set of all these elements which are in U but not in A is called the complement of A and is denoted by A' or A^c

$$\begin{aligned} \therefore A' &= \{x \mid x \in U \text{ but } x \notin A\} \\ &= U - A \end{aligned}$$

E.g. If $U = \{1, 2, 3, \dots, 10\}$

and $A = \{2, 4, 5, 7, 8\}$

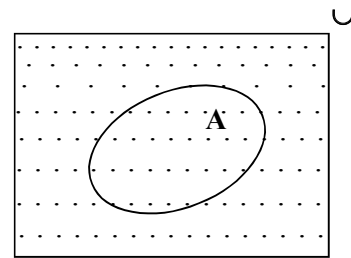
Then $A' = \{1, 3, 6, 9, 10\}$

- Note :** (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$
 (iii) $(A')' = A$ (iv) $U' = \phi$
 (v) $\phi' = U$

3.1.15 VENN DIAGRAM :

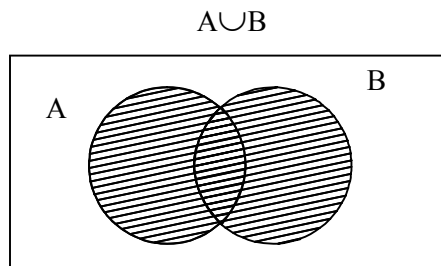
Sometimes pictures are very helpful to make some abstract mathematical ideas concrete. First of all Swiss Mathematician Euler gave an idea to represent a set by the points in a closed curve Later on, British mathematician John-Venn (1834-1883) brought this idea to practice. That is why the diagrams drawn to represent sets are called Venn-Euler diagram or simply Venn Diagram.

In Venn diagram the universal set U is represented by points within a rectangle and any subset A of U is represented by points in a closed circular region within this rectangle. Different set operations like union, intersection etc. may be represented by using Venn-Diagram.

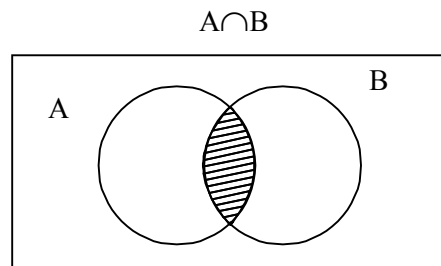


UNION AND INTERSECTION :

The union and intersection of two sets A and B is represented by the shaded region of Fig. 3.1.15(a) and 3.1.15(b) respectively.



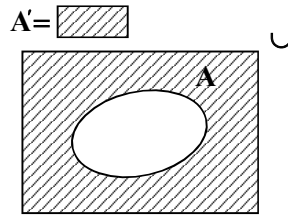
3.1.15(a)



3.1.15(b)

COMPLEMENT :

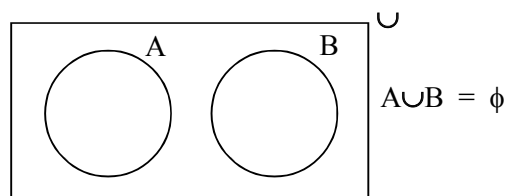
The complement of a set A ie A' or A^c is represented by the shaded region of U outside the region of A [Fig. 3.1.15(c)]



3.1.15(C)

DISJOINT SETS

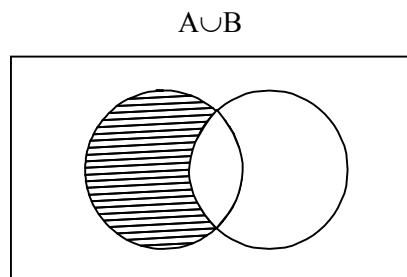
The disjoint sets A and B ie $A \cap B = \phi$ is represented as in Fig.3.1.15(d).



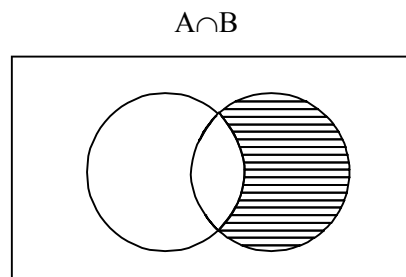
3.1.15(d)

DIFFERENCE :

The difference of A and B ie $A - B$ and difference of B and A ie $B - A$ is represented by Fig.3.1.15(e) and Fig.3.1.15(f) respectively.



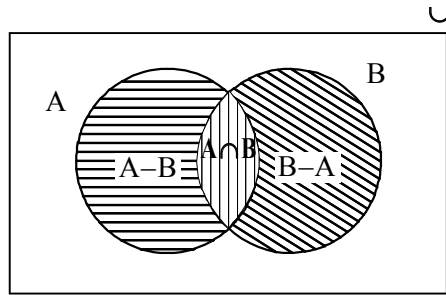
3.1.15(a)



3.1.15(b)

IMPORTANT RESULTS DERIVED FROM VENN DIAGRAM :

Suppose A and B are any two non empty intersecting sets as shown in **Fig.3.1.15(g)**. While counting the elements of $A \cup B$, the elements of $A \cap B$ are counted twice, once in the counting of elements of A and second time in the counting of elements of B.



3.1.15(g)

Thus, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Also, if $A \cap B = \phi$, then

$$n(A \cup B) = n(A) + n(B)$$

It may also be observed from Venn Diagram that :

$$\begin{aligned} \text{(i)} \quad n(A) &= n(A-B) + n(A \cap B) \\ \text{(ii)} \quad n(B) &= n(B-A) + n(A \cap B) \\ \text{(iii)} \quad n(A \cup B) &= n(A-B) + n(A \cap B) + n(B-A) \end{aligned}$$

3.1.16 LAWS OF ALGEBRA OF SETS :

There are certain fundamental laws of algebra of sets which are true for any sets. These laws are given below :

(a) Idempotent Laws :

$$\text{(i)} \quad A \cup A = A \qquad \text{(ii)} \quad A \cap A = A$$

(b) Identity Laws :

$$\begin{aligned} \text{(i)} \quad A \cup \phi &= A & \text{(ii)} \quad A \cap \phi &= \phi \\ \text{(iii)} \quad A \cup U &= U & \text{(iv)} \quad A \cap U &= A \end{aligned}$$

(c) Commutative Law :

$$\text{(i)} \quad A \cup B = B \cup A \qquad \text{(ii)} \quad A \cap B = B \cap A$$

(d) Associative Law :

$$\text{(i)} \quad (A \cup B) \cup C = A \cup (B \cup C) \quad \text{(ii)} \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(e) Distributive Laws :

$$\begin{aligned} \text{(i)} \quad A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ \text{(ii)} \quad A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

(f) De-Morgans Law :

$$(i) (A \cup B)' = A' \cap B'$$

$$(ii) (A \cap B)' = A' \cup B'$$

WORKED OUT EXAMPLES :

Example 5 : Which of the following collections are sets?

- (i) Collection of all even numbers < 50 .
- (ii) Collection of all talented writers of India.
- (iii) Collection of all chairs and tables of your college.
- (iv) Collection of all long rivers of the world.

Solution :

- (i) It is a set
- (ii) It is not a set
- (iii) It is a set
- (iv) It is not a set

as we know that only well-defined collections are set.

Example 6 : Describe the following set in Roaster as well as Rule Method.

- (i) Set of root of the equation
 $x^2 - 7x - 30 = 0$
- (ii) Set of letters of the word 'Mathematics'.
- (iii) Set of prime numbers < 25 .
- (iv) Set of two digits number whose sum of two digits is 7.
- (v) Set of integers whose square is < 30 .

Solution:

$$(i) \text{ Solving } x^2 - 7x - 30 = 0$$

$$\Rightarrow (x-10)(x+3) = 0$$

$$\Rightarrow x = 10 \text{ or } -3$$

$$\therefore A = \{10, -3\}$$

$$= \{x \mid x^2 - 7x - 30 = 0\}$$

$$(ii) A = \{m, a, t, h, e, i, c, s\}$$

$$= \{x \mid x \text{ is a letter of the word 'Mathematics'}\}$$

$$(iii) A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$$= \{x \mid x \text{ is prime and } x < 25\}$$

- (iv) $A = \{16, 61, 25, 52, 34, 43, 70\}$
 $= \{x \mid x \text{ is a two digit number whose sum of two digits is } 7\}$
- (v) $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
 $= \{x \mid x \in \mathbb{I} \text{ and } x^2 < 30\}$

Example 7 : Which of the following set are null? Give reasons.

- (i) Set of prime numbers between 23 and 29.
(ii) Set of positive integers satisfying $x^2 - 8x - 20 = 0$.
(iii) Set of even prime numbers > 2 .
(iv) Set of rational number satisfying $x^2 - 5 = 0$.
(v) $\{x : 5 < x < 6, x \in \mathbb{N}\}$.

Solution :

- (i) Set of prime numbers between 23 and 29 $= \phi$ as all the numbers 24, 25, 26, 27 and 28 lying between 23 and 29 are not prime.
- (ii) Solving $x^2 - 8x - 20 = 0$
 $\Rightarrow (x-10)(x+2) = 0$
 $\Rightarrow x = 10 \text{ or } -2$
 \therefore Set of positive integers satisfying $x^2 - 8x - 20 = 0$ is $\{10\}$ which is not a null set.
- (iii) Set of even prime nos $> 2 = \phi$ as all even nos > 2 are divisible by 2.
- (iv) $x^2 - 5 = 0$
 $\Rightarrow x^2 = 5$
 $\Rightarrow x = \pm \sqrt{5}$ which is not rational
 \therefore Set of rational number satisfying $x^2 - 5 = 0$ is a null set.
- (v) $\{x \mid 5 < x < 6, x \in \mathbb{N}\} = \phi$ as there is no natural number lying between 5 and 6.

Example 8 : Which of the following statement are correct and which are incorrect? Give reasons.

- (i) $\{1, 2, 3\} \subset \{1, 3, 5, 7\}$
(ii) $\{3, 4\} \in \{1, 2, \{3, 4\}, 5\}$
(iii) $\phi \subset A$
(iv) $\{3\} \in \{1, 3, 5\}$
(v) $\{0, u\} \subset \{x : x \text{ is a vowel of English alphabet}\}$.

Solution :

- (i) Incorrect as $2 \notin \{1, 3, 5, 7\}$.
- (ii) Correct as $\{3, 4\}$ is an element of $\{1, 2, \{3, 4\}, 5\}$
- (iii) Correct as it is stated above that ϕ is a proper subset of every set.
- (iv) Incorrect as $\{3\}$ is not an element of $\{1, 3, 5\}$.
- (v) Correct as $\{0, u\} \subset \{a, e, i, o, u\}$ because there are five vowels in English alphabet.

Example 9 : Which of the following pairs of sets are disjoint.

- (i) $\{1, 2, 3, 4\}$ and $\{x \mid x \text{ is a natural number and } 4 \leq x \leq 6\}$.
- (ii) $\{a, e, i\}$ and $\{b, c, d\}$.
- (iii) $A = \{x \mid 5 < x < 9, x \notin \mathbb{N}\}$
 $B = \{x \mid x^2 - 16 = 0\}$.

Solution :

- (i) Given $A = \{1, 2, 3, 4\}$
 Here $B = \{4, 5, 6\}$
 $\therefore A \cap B = \{4\} \neq \phi$
 So, A and B are not disjoint.
- (ii) Here $\{a, e, i\} \cap \{b, c, d\} = \phi$
 They are disjoint sets.
- (iii) $A = \{6, 7, 8\}$
 $B = \{-4, 4\}$
 Since $A \cap B = \phi$
 \therefore A and B are disjoint sets.

Example 10 : If $A = \{3, 5, 7, 9, 11\}$

$$B = \{7, 9, 11, 13\}$$

$$C = \{11, 13, 15\}$$

$$D = \{15, 17\}$$

- Find
- (i) $A \cap C$
 - (ii) $A \cap (B \cup D)$
 - (iii) $(A \cap B) \cup (C \cup D)$
 - (iv) $(A \cup D) \cap (B \cup C)$
 - (v) $(A - B) \cup (C - D)$

Solution : Given $A = \{3, 5, 7, 9, 11\}$

$$B = \{7, 9, 11, 13\}$$

$$C = \{11, 13, 15\}$$

$$D = \{15, 17\}$$

- (i) $A \cap C = \{3, 5, 7, 9, 11\} \cap \{11, 13, 15\}$
 $= \{11\}$
- (ii) $A \cap (B \cup D) = \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13, 15, 17\}$
 $= \{7, 9, 11\}$
- (iii) $(A \cap B) \cup (C \cup D) = \{7, 9, 11\} \cup \{11, 13, 15, 17\}$
 $= \{7, 9, 11, 13, 15, 17\}$
- (iv) $(A \cup D) \cap (B \cup C) = \{3, 5, 7, 9, 11, 15, 17\} \cap \{7, 9, 11, 13, 15\}$
 $= \{7, 9, 11\}$
- (v) $(A - B) \cup (C - D) = \{3, 5\} \cup \{11, 13\}$
 $= \{3, 5, 11, 13\}$

Example 11. If $U = \{a, b, c, d, e, f, g, h\}$

$$A = \{a, c, e, g\}$$

$$B = \{d, e, f, g\}$$

$$C = \{a, f, g, h\}$$

- (i) Find $(A - B)'$
- (ii) Find $A' \cap (B' - C')$
- (iii) Find $(A - B)' \cup (B - C)'$
- (iv) Show that $(A \cap B)' = A' \cup B'$

Solution : Given $U = \{a, b, c, d, e, f, g, h\}$

$$A = \{a, c, e, g\}$$

$$B = \{d, e, f, g\}$$

$$C = \{a, f, g, h\}$$

- (i) $(A - B)' = \{a, c\}' = \{b, d, e, f, g, h\}$
- (ii) Here $(B' - C') = \{a, b, c, h\} - \{b, c, d, e\}$
 $= \{a, h\}$
 $\therefore A' \cap (B' - C') = \{b, d, f, h\} \cap \{a, h\}$
 $= \{h\}$
- (iii) $(A - B)' \cup (B - C)'$ $= \{a, c\}' \cup \{d, e\}'$
 $= \{b, d, e, f, g, h\} \cup \{a, b, c, f, g, h\}$
 $= \{a, b, c, d, e, f, g, h\}$
- (iv) L.H.S. $= (A \cap B)'$
 $= \{e, g\}'$
 $= \{a, b, c, d, f, h\}$

$$\begin{aligned} \text{R.H.S.} &= A' \cup B' \\ &= \{b, d, f, h\} \cup \{a, b, c, h\} \\ &= \{a, b, c, d, f, h\} \end{aligned}$$

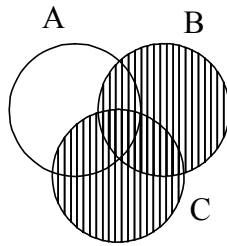
$$\therefore (A \cap B)' = A' \cup B'$$

Example 12 : In a Vonn diagram, shade the following

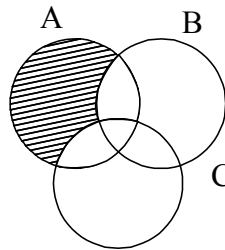
- (i) $A - (B \cup C)$
- (ii) $(A - B)'$
- (iii) $A \subset (B \cap C), B \subset C, C \neq B, C \neq A$

Solution :

(i)

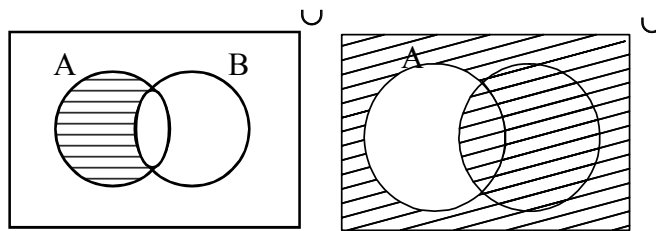


$$B \cup C = \text{[shaded box]}$$



$$\therefore A - (B \cup C) = \text{[shaded box]}$$

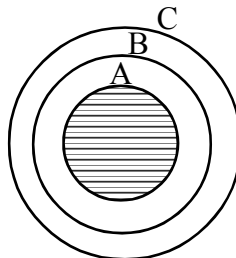
(ii)



$$A - B = \text{[shaded box]}$$

$$(A - B)' = \text{[shaded box]}$$

(iii)



$$A \subset (B \cap C), B \subset C, C \neq B, B \neq A = \text{[shaded box]}$$

Example 13 : In a class there are 35 students who takes Bournvita or Horlicks of these, 25 students take Bournvita and 16 students take Horlicks. How many students takes both Bournvita and Horlicks.

Solution : Let B = Set of students taking Bournvita

H = Set of students taking Horlicks.

$$\therefore n(B) = 25 \quad n(H) = 16$$

and $n(B \cup H) = 35$ [since word 'or' is given]

To find $n(B \cap H)$ [since word 'and' is given]

we have $n(B \cup H) = n(B) + n(H) - n(B \cap H)$

$$\Rightarrow 35 = 25 + 16 - n(B \cap H)$$

$$\Rightarrow n(B \cap H) = 41 - 35 = 6$$

\therefore No of students taking both Bournvita and Horlicks = 6.

Example 14 : A survey conducted in a town having 600 families it was found that 150 families take the Assam Tribune, 225 families take Amar Asom and 100 families take both the newspapers. Find how many families takes

(i) none of the newspapers.

(ii) only the Assam Tribune.

(iii) only the Amar Asom.

Solution : Let A = Set of families who takes the Assam Tribune.

B = Set of families who takes Amar Asom.

$$\therefore n(U) = 600$$

$$n(A) = 150, n(B) = 225, (n(A \cap B) = 100$$

(i) We have

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 150 + 225 - 100 \\ &= 375 - 100 \\ &= 275 \end{aligned}$$

\therefore Families who take none of the newspapers

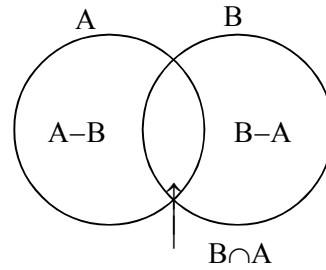
$$\begin{aligned} &= n(A' \cap B') \\ &= n(A \cup B)' \\ &= n(U) - n(A \cup B) \\ &= 600 - 275 \\ &= 325 \end{aligned}$$

(ii) Families who take only the Assam Tribune

$$\begin{aligned} &= n(A-B) \\ &= n(A) - n(A \cap B) \\ &= 150 - 100 \\ &= 50 \end{aligned}$$

(iii) Families who take only Amar Asom

$$\begin{aligned} &= n(B-A) \\ &= n(B) - n(A \cap B) \\ &= 225 - 100 \\ &= 125 \end{aligned}$$



SUMMARY

- ★ A set is a well-defined collection of objects.
- ★ A set which contains no element is called an empty set.
- ★ A set which consists of a definite number of elements is called a finite set, otherwise, the set is called infinite set.
- ★ Two sets A and B are said to be equivalent if they contain the same number of elements.
- ★ Two sets A and B are said to be equal if they contain exactly the same elements.
- ★ A set A is said to be a subset of the set B, if every element of A is also an element of B.
- ★ The power set of a set S, is the collection of all subsets of S and is denoted by P(S).
- ★ The union of two sets A and B is the set of all those elements which belong to either A or B and is denoted by $A \cup B$.
- ★ The intersection of two sets A and B is the set of all those elements which belong to both A and B and is denoted by $A \cap B$.
- ★ The difference of two sets A and B is the set of all those elements which belong to A but not to B and is denoted by $A - B$.
- ★ The complement of a subset A of universal set U is the set of all elements of U which are not the element of A.
- ★ For any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$
- ★ If A and B are any two sets such that,
 - (i) $A \cap B = \phi$, then $n(A \cup B) = n(A) + n(B)$
 - (ii) $A \cap B \neq \phi$, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

EXERCISE 3.1

1. State which of the following collections are sets and which are not? Justify your answer.
 - (i) Teachers of your college.
 - (ii) The collection of all prime numbers < 100 .
 - (iii) The collection of most talented writers of Assam.
 - (iv) The collection of Novel written by Dr. Bhabendra Nath Saikia.
 - (v) The team of eleven best cricket players of Assam.
2. The team following sets using Roaster as well as rule method.
 - (i) Set of letters in the word 'Engineering'.
 - (ii) Set of natural nos whose square is less than 50.
 - (iii) Roots of equation $4x^2 - 8 + 3 = 0$.
 - (iv) Set of prime factors of 24.
 - (v) Set of fractions of the form, $\frac{n}{n+1}$ where n is a natural number.
 - (vi) Set of two digits number whose sum of two digits is 6.
3. State which of the following sets are finite and which are infinite.
 - (i) $\{x : x \in \mathbb{N}, x^2 - 5x - 14 = 0\}$
 - (ii) $\{x : 3x - 1 = 0\}$
 - (iii) $\{x : x \text{ is a real number between } -1 \text{ and } 0\}$
 - (iv) $\{x : x \text{ is a graduate under Gauhati University}\}$
 - (v) All even prime numbers.
 - (iv) Integers which are < 30 .
4. State which of the following sets are equal and which are equivalent
 - A = $\{x : x \text{ is an alphabet of the word 'loyal'}\}$
 - B = $\{x : x \text{ is an alphabet of the word 'wolf'}\}$
 - C = $\{x : x \text{ is an alphabet of the word 'alloy'}\}$
 - D = $\{x : x \text{ is a prime no } < 10\}$
5. Which of the following sets are null?
 - (i) Set of odd natural number divisible by 2.
 - (ii) A = $\{x : x \in \mathbb{N}, x < 5 \text{ and } x > 9\}$

- (ii) $B = \{x : x \text{ is a real root of } x^2+4=0\}$
 (iii) $D = \{x : x+8 = 8\}$
6. Make correct statement by filling in the symbols \subset or \in in the blank spaces.
- (i) $\{4, 5, 6\} \dots\dots \{1, 2, 3, 4, 5, 6\}$
 (ii) $\{c, e, g\} \dots\dots \{a, b, c, d, e\}$
 (iii) $\{x : x \text{ is a student of H.S. 2nd year}\} \dots\dots \{x : x \text{ is a student of B.Com 3rd year}\}$
 (iv) $\{x : x \text{ is a prime number}\} \dots\dots \{x : x \text{ is an integer}\}$
 (v) $\{x : x \text{ is a square}\} \dots\dots \{x : x \text{ is a rhombus}\}$
 (vi) $\{x : x \text{ is an equilateral triangle}\} \dots\dots \{x : x \text{ is a right angled triangle}\}$
 (vii) $\{x : x \text{ is a circle}\} \dots\dots \{x : x \text{ is a circle of radius 2 unit}\}$
7. Which of the following statements are correct and which are incorrect.
- (i) $\{2\} \subset \{1, 2, 4\}$
 (ii) $\{a, b\} \subset \{x \mid x \text{ is a vowel of English alphabet}\}$
 (iii) $\{1, 4\} \in \{1, 4, 6, 7\}$
 (iv) $\{x : x + 8 = 8\} = \phi$
 (v) $\{a\} \in \{a, b, \{a\}, \{b\}\}$
 (vi) $\{\phi\} \subset \{p, q, s\}$
8. Which of the following statements are incorrect and why?
 given $A = \{a, b, \{c, d\}, e\}$
- (i) $\{c, d\} \subset A$
 (ii) $\{c, d\} \in A$
 (iii) $\{a, b, e\} \in A$
 (iv) $\phi \subset A$
 (v) $a \subset A$
 (vi) $\{\{c, d\}, e\} \subset A$
9. Which of the following set is a singleton set?
- (i) $\{x : x \in \mathbb{R}, 2x^2 - x = 0\}$
 (ii) $\{x : x \in \mathbb{N}, 7x = 9\}$
 (iii) $\{x : x^2 = 36\}$
 (iv) $\{x : 2x^2 - 9x - 5 = 0, x \in \mathbb{N}\}$
10. If $A = \{x : x \text{ is an integer, } 1 \leq x \leq 12\}$
 $B = \{x : x \text{ is a prime no } < 20\}$
 $C = \{x : x \text{ is an odd natural number } < 15\}$

Find (i) $A \cap B \cap C$

(ii) $A - (B \cup C)$

(iii) $B \Delta C$

(iv) $C - B$

(v) $(A \cap B) \cup (A \cap C)$

(vi) $(A - B) \cup (B - C)$

11. Which of the following pair of sets are disjoint.

(i) $A = \{x : x \in \mathbb{N}, 5 < x < 9\}$

$B = \{x : x^2 - 4x - 12 = 0\}$

(ii) $L = \{x : x \text{ is a prime no } \leq 5\}$

$M = \{x : x^2 - 16x + 63 = 0\}$

(iii) $C = \{x : x \text{ is a vowel of English alphabet}\}$

$D = \{x : x \text{ is a one of the last five letters of English alphabet}\}$

12. Find $A \cap B$ in each of the following cases.

(i) $A = \{c, d, e, f\}$ $B = \{d, f, g, h\}$

(ii) $A = \{x : x \text{ is an even no } < 8\}$

$B = \{x : x \text{ is an even prime number}\}$

(iii) $A = \{x : 5 < x < 10\}$

$B = \{x : 4 < x < 8\}$

13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set and

$A = \{2, 3, 5, 7\}$

$B = \{1, 4, 7, 8\}$

$C = \{2, 3, 6, 8, 9\}$

Find (i) $B \cap (A - C)$

(ii) $A' \cap (B - C)'$

(iii) $A' \cap (B \Delta C)$

(iv) $B' \cap (A' - C')$

14. If $A = \{a, b, c, d\}$

$B = \{b, d, f, h\}$

$C = \{c, d, e, f\}$

Verify that

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) $A - (A - B) = A \cap B$

(iv) $A \cap (B - C) = (A \cap B) - (B \cap C)$

15. If S = Set of all boarders in a hostel.

F = Set of all boarders who takes fish.

M = Set of all boarders who takes meat.

and G = Set of all boarders who takes egg.

Express the following set in language.

(i) F' (ii) $(M \cap G)'$ (iii) $F \cup M$

(iv) $M - G$ (v) $S - (M \cup G)$

16. Verify the following using Venun Diagram.

(i) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

(ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) $(A \cup B) = A \cap B$

17. There are 100 students in a class who takes at least one of the subjects commercial Mathematics or Economics. If 75 students take commercial Mathematics and 60 students take Economics. Find the number of students who have taken.

(i) Both the subjects.

(ii) Only Commercial Mathematics.

(iii) Only Economics.

18. In a club of 100 members, 40 do not like to play football, 30 do not like to play volleyball and 45 like to play both the game. Find the number of members who

(i) like to play only football

(ii) like to play only volleyball

(iii) do not like to play none of the games.

19. Out of 150 invitios in a party, 90 prefer only apple juice, 20 prefer only mango juice and 30 has taken both the juice. Find the number of invitios who have taken none of the two juice.

20. In a group, 100 know Assamese, 50 know English and 25 know both the language. If each of the members know at least one of the two languages, find the total number of members in the group.

ANSWERS 3.1

1. (i), (ii), (iv) are set.
 (iii), (v) not set.
2. (i) {e, n, g, i, r}
 $\{x \mid x \text{ is a letter of the word 'Engineering'}\}$
 (ii) {1, 2, 3, 4, 5, 6, 7}
 $\{x : x \in \mathbb{N}, x^2 < 50\}$
 (iii) $\left\{\frac{1}{2}, \frac{3}{2}\right\}$
 $\{x : x \text{ is a root of the equation } 4x^2 - 8x + 3 = 0\}$
 (iv) {2, 3}
 $\{x \mid x \text{ is a prime factor of } 24\}$
 (v) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$
 $\left\{x : x = \frac{n}{n+1}, n \in \mathbb{N}\right\}$
 (vi) {15, 51, 24, 42, 60, 33}
 $\{x : x = 10k + b \text{ where } l + k = 6, l, k \in \mathbb{N}\}$
3. Finite \rightarrow (i), (ii), (iv), (v)
 Infinite \rightarrow (iii), (vi)
4. $A = \{l, o, y, a\}$
 $B = \{w, o, l, f\}$
 $C = \{a, l, o, y\}$
 $D = \{2, 3, 5, 7\}$
 $\therefore A = C, A \sim B, A \sim C, A \sim D$
 $B \sim C, B \sim D, C \sim D$
5. Null set \rightarrow (i), (ii), (iii)
6. (i) \subset (ii) $\not\subset$ (ii) $\not\subset$
 (iv) \subset (v) \subset (vi) $\not\subset$ (vii) $\not\subset$

7. Correct \rightarrow (i), (v)
Incorrect \rightarrow (ii), (iii), (iv), (vi)
8. (i) Incorrect as $\{c, d\} \in A$
(iii) Incorrect as $\{a, b, c\} \subset A$
(v) Incorrect as $a \in A$
9. (ii), (iv)
10. (i) $\{3, 5, 7, 11\}$
(ii) $\{4, 6, 8, 10, 12\}$
(iii) $\{1, 2, 9, 17, 19\}$
(iv) $\{1, 9\}$
(v) $\{1, 2, 3, 5, 7, 9, 11\}$
(vi) $1, 2, 4, 6, 8, 9, 10, 12, 17, 19\}$
11. (ii) $L \cap M = \phi$, L and M are disjoint
(iii) $C \cap D = \phi$, C and D are disjoint
12. (i) $\{d, f\}$
(ii) $\{2\}$
(iii) $\{x \mid 5 < x < 8\}$
13. (i) $\{7\}$
(ii) $\{6, 8, 9\}$
(iii) $1, 4, 6, 9\}$
(iv) $\{6, 9\}$
14. (i) Set of boarders who does not take fish.
(ii) Set of boarder who does not take both meat and egg.
(iii) Set of boarders who take both fish and meat.
(iv) Set of boarder who take meat but not egg.
(v) Set of boarders who takes neither meat nor egg.
17. (i) 35, (ii) 40, (iii) 25
18. (i) 15, (ii) 25, (iii) 15
19. 10
20. 125

3.2 DETERMINANTS

3.2.1 INTRODUCTION

The theory of determinants had its origin in the study of system of simultaneous linear equation, but it has many other applications as well. Many complicated expressions which involve lengthy calculations may be solved easily if they are expressed as determinants.

Let us consider the following equations :

$$a_1x + b_1 y = d_1 \dots\dots\dots(1)$$

$$a_2 x + b_2 y = d_2 \dots\dots\dots(2)$$

If $a_1b_2 - a_2b_1 \neq 0$, solving (1) and (2) we get

$$x = \frac{b_2d_1 - b_1d_2}{a_1b_2 - a_2b_1}$$

and $y = \frac{a_1d_2 - a_2d_1}{a_1b_2 - a_2b_1}$

The algebraic expression $(a_1b_2 - a_2b_1)$ is expressed in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is called a determinant of second order. It has two rows and two columns. The elements in the horizontal direction form rows and those in the vertical direction form columns.

1st column	2nd column	
$\begin{vmatrix} a_1 \\ a_2 \end{vmatrix}$	$\begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$	\rightarrow 1st row \rightarrow 2nd row

The numbers a_1, a_2, b_1, b_2 which form the determinant are called the elements of the determinant. a_1b_2, a_2b_1 are called the terms and the expression $a_1b_2 - a_2b_1$ is called the expansion of the determinant.

The elements which lies on the line joining the 1st element and the last element of the determinant are called diagonal elements.

The diagonals on which those diagonal elements lie is called Principal (or leading) diagonal.

e.g. In $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \rightarrow$ Principal Diagonal

a_1 and b_2 are diagonal elements.

- Note :** (i) There are always equal number of rows and columns in a determinant.
(ii) A determinant is usually denoted by D or Δ .

3.2.2 EXPANSION OF A DETERMINANT OF SECOND ORDER

It is seen that the second order determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_2a_1$$

shows that,

Expansion of a second order determinant = Product of the terms of leading diagonal-
Product of the remaining two diagonal elements.

Note : A second order determinant has 2^2 elements and give on expansion 2 terms.

$$\text{e.g. } \begin{vmatrix} 4 & 5 \\ -6 & 2 \end{vmatrix} = 4 \times 2 - 5 \times (-6) \\ = 8 + 30 = 38$$

3.2.3 THIRD ORDER DETERMINANT

If we have nine numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ and we arrange them in 3 rows and 3 columns like

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

it is called a determinant of order 3. A third order determinant has 3^2 elements and give on expansion 3 terms.

- Note :** (i) So, in general n^2 numbers are arranged in n rows and n columns, it is said to be a determinant of order n which on expansion gives n terms.
(ii) Here we will restrict our discussion to determinant of order 2 and 3 only.

3.2.4 MINOR AND COFACTORS

The minor of any element in a determinant Δ is the determinant obtained by omitting the row and column to which the element belongs. Therefore in the determinant Δ of order 3

$$\text{ie } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{minor of } a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$$

$$\text{minor of } b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$$

The minor of any element in a 3rd order determinant is thus a second order determinant.

The co-factor of any element in a determinant is equal to the corresponding minor with a proper sign.

Co-factor of any element

$= (-1)^{R+C} \times$ minor of that element where R and C denotes the row and column in which that particular element exists.

The co-factor of any element is generally denoted by the corresponding capital letter; for instance, the co-factor of a_1 is A_1 , b_2 is B_2 etc.

$$\therefore \text{Co-factor of } a_1 = A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$= + (b_2c_3 - b_3c_2)$$

$$\text{Co-factor of } b_2 = B_2 = (-1)^{2+2} \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$= + (a_1c_3 - a_3c_1)$$

$$\text{Co-factor of } c_3 = C_3 = (-1)^{3+3} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= + (a_1b_2 - a_2b_1)$$

3.2.5 EXPANSION OF A DETERMINANT OF ORDER THREE

The rule for expansion of a third order determinant is as follows :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

$$= a_1 A_1 + b_1 B_1 + C_1 C_1$$

where A_1 , B_1 and C_1 are cofactors of the elements a_1 , b_1 and c_1 respectively.

Here we have expanded the determinant **in terms of the elements of the 1st Row**.

Expanding the determinant in terms of the element of first column we get–

$$\begin{aligned} & a_1A_1 + a_2A_2 + a_3A_3 \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &= a_1(b_2c_3 - b_2c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= \Delta \end{aligned}$$

\therefore Expansion of a third order determinant = Sum of the products of the elements of any row or column and the corresponding co-factors of the elements.

e. g. Expand $\begin{vmatrix} 4 & 3 & -6 \\ 1 & -2 & 5 \\ 0 & -7 & 2 \end{vmatrix}$

$$\begin{aligned} &= 4 \begin{vmatrix} -2 & 5 \\ -7 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} + (-6) \begin{vmatrix} 1 & -2 \\ 0 & -7 \end{vmatrix} \\ &= 4(-4 + 35) - 3(2-0) - 6(-7-0) \\ &= 124 - 6 + 42 \\ &= 160 \end{aligned}$$

3.2.6 PROPERTIES OF DETERMINANTS

Now we are going to study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros, in a row or column. These properties are true for determinants of any order. However, we shall restrict ourselves upto determinants of third order only.

Property 1 : The value of a determinant remain unchanged if its rows and columns are interchanged.

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along 1st Row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

By interchanging rows and columns of Δ we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding along 1st column we get,

$$\Delta_1 = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

$$\text{Hence, } \Delta = \Delta_1$$

Note : This property is symbolically written as $R \leftrightarrow C$

Property 2 : If any two rows (or columns) of a determinant are interchanged, then sign of the determinant changes.

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

Interchanging first and third row, the new determinant is given by

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row we get

$$\begin{aligned} \Delta_1 &= a_1 (c_2 b_3 - c_3 b_2) - a_2 (c_1 b_3 - c_3 b_1) + a_3 (c_1 b_2 - c_2 b_1) \\ &= - [a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)] \end{aligned}$$

$$\text{Clearly, } \Delta_1 = -\Delta$$

Similarly, we can verify the result by interchanging any two columns.

Note : This property is symbolically written as –

$$R_i \leftrightarrow R_j, \quad \text{if } i^{\text{th}} \text{ and } j^{\text{th}} \text{ row are interchanged.}$$

$$C_i \leftrightarrow C_j, \quad \text{if } i^{\text{th}} \text{ and } j^{\text{th}} \text{ column are interchanged.}$$

Property 3 : If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

If we interchange R_1 and R_3 , by property 2,

$$\Delta = -\Delta$$

$$\Rightarrow 2\Delta = 0$$

$$\Rightarrow \Delta = 0$$

Property 4 : If each element of a row (or column) of a determinant is multiplied by a constant k , then its value gets multiplied by k .

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and Δ_1 be the determinant obtained by multiplying the elements of the first row by k .

$$\text{Then } \Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along 1st row, we get

$$\begin{aligned} \Delta_1 &= ka_1 (b_2c_3 - b_3c_2) - kb_1 (a_2c_3 - a_3c_2) + kc_1 (a_2b_3 - a_3b_2) \\ &= k [a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)] \\ &= k\Delta \end{aligned}$$

$$\text{Hence } \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remark : By this property we can take out any common factor from any one row or any one column of a given determinant.

Property 5 : If some or all elements of a row (or column) of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 + \lambda_1 & b_1 & c_1 \\ a_2 + \lambda_2 & b_2 & c_2 \\ a_3 + \lambda_3 & b_3 & c_3 \end{vmatrix}$$

Expanding Δ in terms of the elements of first column we get,

$$\begin{aligned} \Delta &= (a_1 + \lambda_1)(b_2 c_3 - b_3 c_2) - (a_2 + \lambda_2)(b_1 c_3 - b_3 c_1) + (a_3 + \lambda_3)(b_1 c_2 - b_2 c_1) \\ &= [a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)] + [\lambda_1(b_2 c_3 - b_3 c_2) \\ &\quad - \lambda_2(b_1 c_3 - b_3 c_1) + \lambda_3(b_1 c_2 - b_2 c_1)] \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & b_1 & c_1 \\ \lambda_2 & b_2 & c_2 \\ \lambda_3 & b_3 & c_3 \end{vmatrix} \end{aligned}$$

Property 6 : If each element of a row (or column) of a determinant is multiplied by a real number k and the resulting product is added to the corresponding entry of another row (or column), the resulting determinant is equal to the original determinant.

Verification :

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and } \Delta_1 = \begin{vmatrix} a_1 & b_1 + ka_1 & c_1 \\ a_2 & b_2 + ka_2 & c_2 \\ a_3 & b_3 + ka_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix} \quad [\text{By property 5}]$$

$$\begin{aligned}
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} \quad [\text{By property 4}] \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \cdot 0 \quad [\because c_2 = c_1] \\
&= \Delta
\end{aligned}$$

Note : Symbolically this property is expressed as –

$$\begin{aligned}
R_i &\rightarrow R_i + m R_j \\
\text{or } C_i &\rightarrow C_i + m C_j
\end{aligned}$$

i.e. m times of j^{th} row is added to i^{th} row and i^{th} row is changed.

WORKED OUT EXAMPLE :

Example 1: Evaluate $\begin{vmatrix} 3 & -4 & 5 \\ 6 & 7 & 0 \\ 2 & 8 & -3 \end{vmatrix}$

$$\begin{aligned}
&= 3 \begin{vmatrix} 7 & 0 \\ 8 & -3 \end{vmatrix} - (-4) \begin{vmatrix} 6 & 0 \\ 2 & -3 \end{vmatrix} + 5 \begin{vmatrix} 6 & 7 \\ 2 & 8 \end{vmatrix} \\
&= 3(-21-0) + 4(-18-0) + 5(48-14) \\
&= -63-72+170 = 35
\end{aligned}$$

Example 2:

Find the value of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

Solution :

$$\begin{aligned}
\text{Given } \begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \\
\Rightarrow 3-x^2 &= 3-8 \\
\Rightarrow x^2 &= 8 \\
\Rightarrow x &= \pm 2\sqrt{2}
\end{aligned}$$

Example 3:

Without expanding prove that

$$(i) \quad \begin{vmatrix} 2 & 4 & -10 \\ -3 & 9 & 15 \\ 7 & -2 & -35 \end{vmatrix} = 0$$

Solution :

$$\begin{aligned} \text{L.H.S.} &= \begin{vmatrix} 2 & 4 & -10 \\ -3 & 9 & 15 \\ 7 & -2 & -35 \end{vmatrix} \\ &= -5 \begin{vmatrix} 2 & 4 & 2 \\ -3 & 9 & -3 \\ 7 & -2 & 7 \end{vmatrix} \quad (\text{Taking } -5 \text{ common from } c_3) \\ &= -5 \times 0 \quad [\because c_1 = c_3] \\ &= 0 = \text{R.H.S.} \end{aligned}$$

$$(ii) \quad \begin{vmatrix} 3 & -6 & 7 \\ 7 & -2 & 11 \\ 1 & -8 & 5 \end{vmatrix} = 0$$

$$\begin{aligned} \text{L.H.S.} &= \begin{vmatrix} 3 & -6 & 7 \\ 7 & -2 & 11 \\ 1 & -8 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -6 & 7 \\ 4 & 4 & 4 \\ -2 & -2 & -2 \end{vmatrix} \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix} \\ &= 4 \times (-2) \begin{vmatrix} 3 & -6 & 7 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \quad (\text{Taking } 4 \text{ and } -2 \text{ common from } R_2 \text{ and } R_3) \\ &= -8 \times 0 \quad (\because R_2 = R_3) \\ &= 0 = \text{R.H.S.} \end{aligned}$$

$$(iii) \begin{vmatrix} 1 & p & q+r \\ 1 & q & r+p \\ 1 & r & p+q \end{vmatrix} = 0$$

$$\text{L.H.S.} = \begin{vmatrix} 1 & p & q+r \\ 1 & q & r+p \\ 1 & r & p+q \end{vmatrix}$$

$$= \begin{vmatrix} 1 & p & p+q+r \\ 1 & q & p+q+r \\ 1 & r & p+q+r \end{vmatrix} c_3 \rightarrow c_3 + c_2$$

$$= (p+q+r) \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} \quad [\text{Taking } (p+q+r) \text{ common from } c_3]$$

$$= (p+q+r) \times 0 \quad [\because c_1 = c_3]$$

$$= 0 = \text{R.H.S.}$$

4. Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -y \\ 0 & x & -y \\ 1 & 1 & 1+y \end{vmatrix} \begin{array}{l} R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 - R_3 \end{array}$$

$$= -y \begin{vmatrix} 0 & x \\ 1 & 1 \end{vmatrix}$$

$$= -y(0-x)$$

$$= xy = \text{R.H.S.}$$

5. Prove that
$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

$$\text{L.H.S.} = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad (\text{Taking } a, b, c \text{ common from } R_1, R_2 \text{ and } R_3)$$

$$= (abc)(abc) \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad (\text{Taking } a, b, c \text{ common from } C_1, C_2 \text{ and } C_3)$$

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & +2 & -1 \end{vmatrix} \begin{array}{l} C_1 \rightarrow C_1 + C_3 \\ C_2 \rightarrow C_2 - C_3 \end{array}$$

$$= a^2b^2c^2 (4-0)$$

$$= 4a^2b^2c^2 = \text{R.H.S.}$$

6. Show that
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)^3$$

$$\text{L.H.S.} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \begin{array}{l} R_1 \rightarrow R_1 + R_2 + R_3 \end{array}$$

$$\begin{aligned}
&= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad [\text{Taking } (a+b+c) \text{ common from } R_1] \\
&= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix} \begin{array}{l} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \end{array} \\
&= (a+b+c) \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix} \\
&= (a+b+c) [(-1)^2(a+b+c)^2 - 0] \\
&= (a+b+c)^3 \\
&= \text{R.H.S.}
\end{aligned}$$

7. Solve $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$

Solution : Given $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x+2 & 1 & 1 \\ x+2 & x & 1 \\ x+2 & 1 & x \end{vmatrix} = 0 \quad C_1 \rightarrow C_1 + C_2 + C_3$$

$$\Rightarrow (x+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0 \quad [\text{Taking } (x+2) \text{ common from } C_1]$$

$$\Rightarrow (x+2) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 1 & 0 & x-1 \end{vmatrix} = 0 \quad \begin{array}{l} C_2 \rightarrow C_2 - C_1 \\ C_3 \rightarrow C_3 - C_1 \end{array}$$

$$\Rightarrow (x+2) (x-1)^2 = 0$$

$$\Rightarrow (x+2) = 0 \text{ or } (x-1)^2 = 0$$

$$\Rightarrow x = -2 \text{ or } x = 1, 1$$

$$\therefore x = -2, 1, 1 \text{ Ans.}$$

3.2.6 SOLUTION OF EQUATIONS USING CRAMER'S RULE

Gabriel Cramer obtained a simple method of solution of linear simultaneous equations by the principle of determinants.

Let us consider the three simultaneous equations :

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

where $\Delta_x, \Delta_y, \Delta_z$ are the determinants obtained from Δ by substituting the column (d_1, d_2, d_3) for first, second and third columns respectively.

$$\text{Now, } \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

$$= x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + z \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$= x.\Delta + y.0 + z.0 \text{ [By property 4]}$$

$$\Rightarrow \Delta_x = x.\Delta$$

$$\Rightarrow x = \frac{\Delta_x}{\Delta}$$

$$\text{Similarly, } y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

Thus according to Cramer's Rule.

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{1}{\Delta}; \quad \Delta \neq 0$$

Note : (i) If $\Delta \neq 0$, the system is consistent and has a unique solution.

(ii) If $\Delta = 0$ and at least one of $\Delta_x, \Delta_y, \Delta_z \neq 0$, then the system has no solution and is inconsistent.

Remark : Here we will concentrate only to solution of those equations where $\Delta \neq 0$.

WORKED OUT EXAMPLE :

Example 8 : Solve by Cramer's Rule.

$$2x+3y = 13$$

$$x + 7y = 23$$

Solution : Here $\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} = 14 - 3 = 11 \neq 0$

$$\Delta_x = \begin{vmatrix} 13 & 3 \\ 23 & 7 \end{vmatrix} = 91 - 69 = 22$$

$$\Delta_y = \begin{vmatrix} 2 & 13 \\ 1 & 23 \end{vmatrix} = 46 - 13 = 33$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{22}{11} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{33}{11} = 3$$

Example 9: Solve by Cramer's Rule.

$$x+2y+3z-6 = 0$$

$$2x+4y-7 = -z$$

$$3x+2y+9z-14 = 0$$

Solution :

[**Note :** It is to be noted that before forming the determinants. We have to transpose all the constant terms to Right hand side.]

Here $x + 2y + 3z = 6$

$$2x + 4y + z = 7$$

$$3x + 2y + 9z = 14$$

$$\begin{aligned} \text{Here } \Delta &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} \\ &= 1(36-2) - 2(18-3) + 3(4-12) \\ &= 34-30-24 = -20 \neq 0 \end{aligned}$$

$$\begin{aligned} \Delta_x &= \begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} \\ &= 6(36-2) - 2(63-14) + 3(14-56) \\ &= 204-98-126 = -20 \end{aligned}$$

$$\begin{aligned} \Delta_y &= \begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} \\ &= 1(63-14) - 6(18-3) + 3(28-21) \\ &= 49-90+21 = -20 \end{aligned}$$

$$\begin{aligned} \Delta_z &= \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} \\ &= 1(56-14) - 2(28-21) + 6(4-12) \end{aligned}$$

$$= 42 - 14 - 48$$

$$= -20.$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-20}{-20} = 1$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-20}{-20} = 1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-20}{-20} = 1$$

Ans : $x = 1, y = 1, z = 1.$

Example 10 : Solve $3x + y + z - 10 = 0$

$$x + y - z = 0$$

$$5x - 9y = 1$$

Solution : Here, $3x + y + z = 10$

$$x + y - z = 0$$

$$5x - 9y + 0z = 1$$

$$\begin{aligned} \therefore \Delta &= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 5 & -9 & 0 \end{vmatrix} \\ &= 3(0-9) - 1(0+5) + 1(-9-5) \\ &= -27 - 5 - 14 \\ &= -46 \end{aligned}$$

$$\begin{aligned} \Delta_x &= \begin{vmatrix} 10 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -9 & 0 \end{vmatrix} \\ &= 10(0-9) - 1(0+1) + 1(0-1) \\ &= -90 - 1 - 1 \\ &= -92 \end{aligned}$$

$$\Delta_y = \begin{vmatrix} 3 & 10 & 1 \\ 1 & 0 & -1 \\ 5 & 1 & 0 \end{vmatrix}$$

$$\begin{aligned}
 &= 3(0+1) - 10(0+5) + 1(1-0) \\
 &= 3 - 50 + 1 \\
 &= -46
 \end{aligned}$$

$$\begin{aligned}
 \Delta_z &= \begin{vmatrix} 3 & 1 & 10 \\ 1 & 1 & 0 \\ 5 & -9 & 1 \end{vmatrix} \\
 &= 3(1-0) - 1(1-0) + 10(-9-5) \\
 &= 3 - 1 - 140 \\
 &= -138
 \end{aligned}$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-92}{-46} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-46}{-46} = 1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-138}{-46} = 3$$

Ans : $x=2, y=1, z=3$.

SUMMARY

$$* \quad |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$* \quad |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

- * Minor M_{ij} of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which the element a_{ij} lies.
- * Minor of an element of a determinant of order n ($n \geq 2$) is a determinant of order $n-1$.
- * Co-factor A_{ij} of an element a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$
- * If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

- * If elements of a row (or column) are multiplied with their respective cofactors, then their sum is equal to the value of the determinant.
- * If the rows are changed into columns and columns into rows, the value of the determinant remains the same.
- * If we interchange any two rows (or columns), then sign of the determinant changes.
- * If any two rows (or columns) are identical, then value of the determinant is zero.
- * If we multiply each element of a row (or column) of a determinant by a non zero real number k , then the value of the determinant is multiplied by k .
- * If elements of a row (or column) in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.
- * If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (or columns) are added, then the value of the determinant remains the same.
- * If $a_1x + b_1y + c_1z = d_1$
 $a_2x + b_2y + c_2z = d_2$
 $a_3x + b_3y + c_3z = d_3$

$$\text{and } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then by Cramer's rule

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

EXERCISE 3.2

1. (i) Write down the minors of -2 and 4 in $\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{vmatrix}$

(ii) Write down the co-factors of 3 and -5 in $\begin{vmatrix} 1 & 0 & -5 \\ 3 & -1 & 2 \\ 4 & 6 & -3 \end{vmatrix}$

2. Write down co-factors of elements of the second row in $\begin{vmatrix} 5 & -2 & 0 \\ 6 & 1 & -3 \\ -4 & 2 & 1 \end{vmatrix}$

3. Write down the minor and co-factors of the elements of first column in each of the following determinant

(i) $\begin{vmatrix} 5 & 17 \\ 0 & -1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

4. Evaluate $\begin{vmatrix} 5 & 1 & 0 \\ 2 & 3 & -1 \\ -3 & 2 & 0 \end{vmatrix}$

5. Without expanding prove that each of the determinant given below is equal to zero

(i) $\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

(ii) $\begin{vmatrix} 3 & 2 & -4 \\ 9 & 6 & -12 \\ 31 & 16 & 19 \end{vmatrix}$

(iii) $\begin{vmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 10 & 13 & 16 \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 1^2 & 2^2 \\ 2 & 2^2 & 4^2 \\ 3 & 3^2 & 6^2 \end{vmatrix}$

$$(v) \begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & b & ca \\ \frac{1}{c} & c & ab \end{vmatrix}$$

$$(vi) \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$$

$$(vii) \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

6. Solve for x

$$(i) \begin{vmatrix} x & 0 & 0 \\ 2 & 5 & 7 \\ 0 & 1 & 2 \end{vmatrix} = 9$$

$$(ii) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

$$(iii) \begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$$

7. Prove without expanding

$$(i) \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} 1 & x & x^2 - yz \\ 1 & y & y^2 - zx \\ 1 & z & z^2 - xy \end{vmatrix} = 0$$

$$(iii) \begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}$$

8. Prove that

$$(i) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

$$(ii) \begin{vmatrix} 1+x & y & z \\ x & 1+y & z \\ x & y & 1+z \end{vmatrix} = 1+x+y+z$$

$$(iii) \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2 \quad (iv) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(v) \begin{vmatrix} x+y & z & z-x \\ y+z & x & x-y \\ z+x & y & y-z \end{vmatrix} = x^3 + y^3 + z^3 - 3xyz \quad (vi) \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$$

$$(vii) \begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} \quad (viii) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

$$= a_1a_2a_3 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)$$

9. Solve

$$(i) \begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0 \quad (ii) \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$$

10. Solve by Cramer's Rule

$$(i) \begin{cases} 3x+4y = 2 \\ 9x-16y = -1 \end{cases} \quad (ii) \begin{cases} 2x+3y = 5 \\ 3x-2y = 1 \end{cases}$$

$$(iii) \begin{cases} x+2z = 7 \\ 3x+4y = 11 \\ 3y-5z = -9 \end{cases} \quad (iv) \begin{cases} x_1+3x_2+5x_3 = 22 \\ 5x_1-3x_2+2x_3 = 5 \\ 9x_1+8x_2-3x_3 = 16 \end{cases}$$

$$(v) \begin{cases} \frac{2}{x} - \frac{1}{y} = 8 \\ \frac{3}{x} + \frac{2}{y} = 5 \end{cases}$$

$$(vi) \begin{cases} x-y = 1 \\ x+z = -6 \\ x+y-2z = 3 \end{cases}$$

ANSWERS 3.2

1. (i) Minor = 5; -5
(ii) Co-factors = 30; 22
2. Co-factors of 2nd row elements = 2, 5, -2
3. (i) Minors = -1, 17; Co-factors = -1, -17
(ii) Minors = ab^2-ac^2 ; a^2b-bc^2 ; a^2c-b^2c
Co-factors = ab^2-ac^2 ; bc^2-a^2b ; a^2c-b^2c
4. 13
5. (i) $x = 3$; (ii) $x = 1, 1$; (iii) $x = 2, -\frac{17}{7}$
9. (i) $x = 0, 9$ or -9
(ii) $x = 0$ or $-(a+b+c)$
10. (i) $x = \frac{1}{3}, y = \frac{1}{4}$
(ii) $x = 1, y = 1$
(iii) $x = 1, y = 2, z = 3$
(iv) $x_1 = 1, x_2 = 2, x_3 = 3$
(v) $x = \frac{1}{3}, y = -\frac{1}{2}$
(vi) $x = -2, y = -3, z = -4$

3.3 MATRIX

3.3.1 INTRODUCTION

It is often desirable to present a set of numbers or other elements in a rectangular array (Impressive display). In such an array it is conventional to call the vertical lines column and the horizontal lines rows.

Suppose we wish to express the information – Papori has purchased 10kg of rice, 5kg of atta and 3kg of sugar and Rupali has purchased 7kg of rice, 4kg of atta and 5kg of sugar in a compact manner in a rectangular array. This may be expressed as follows :

	Rice	Atta	Sugar	
Papori	10	5	3	→ First Row
Rupali	7	4	5	→ Second Row
	↑	↑	↑	
	First	Second	Third	
	Column	Column	Column	

or

	Papori	Rupali	
Rice	10	7	→ First Row
Atta	5	4	→ Second Row
Sugar	3	5	→ Third Row
	↑	↑	
	First	Second	
	Column	Column	

Such an array when subjected to certain rules of operation is called a Matrix.

The knowledge of matrix is necessary in various branches of mathematics. Matrix is one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent. For presenting large data in a compact form in areas like budgeting, sales projection, analysis of results in any experiment, for solution of simultaneous linear equations, for coding and decoding of secret messages, for solving problems of linear programming, the theory of matrix has been constantly applied.

In this chapter, we shall become acquainted with the fundamentals of matrix and matrix algebra.

3.3.2 DEFINITION

A matrix (Plural is matrices) is an ordered rectangular array of real numbers (or other suitable entities), arranged in rows and columns.

We denote matrix by capital letters and the entities are enclosed by a pair of brackets either [] or ().

The following are some examples of matrices :

$$A = \begin{bmatrix} -4 & 6 \\ 0 & \sqrt{2} \\ 1 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 8 & -7 & 6 \\ 0 & 1 & -2 \\ 3 & 5 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 2+x & x^3 & 4-x \\ 5 & 3 & 2x+1 \end{bmatrix}$$

As mentioned earlier, the horizontal lines are called rows and vertical lines are called columns.

Thus matrix A has 3 rows and 2 columns.

Matrix B has 3 rows and 3 columns.

Matrix C has 2 rows and 3 columns.

3.3.3 ELEMENT OF A MATRIX

The entities are called the elements of a matrix and is denoted by a_{ij} which means it is the element lying in i^{th} row and j^{th} column.

So referring to the above example

$$\begin{array}{lll} A, & a_{11} = -4 & a_{21} = 0 & a_{31} = 1 \\ & a_{12} = 6 & a_{22} = \sqrt{2} & a_{32} = -8 \end{array}$$

3.3.4 ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as m by n matrix). So, referring to the above examples, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix.

A matrix of order $m \times n$ has mn elements.

We observe that matrix A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements respectively.

In general, a $m \times n$ matrix has the following rectangular array :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1j} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2j} \cdots a_{2n} \\ \vdots & & \\ a_{i1} & a_{i2} & a_{i3} \cdots a_{ij} \cdots a_{in} \\ \vdots & \vdots & \vdots \quad \vdots \quad \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mj} \cdots a_{mn} \end{bmatrix}$$

$m \times n$

The above matrix may be written in a more concise manner as

$$A = [a_{ij}]_{m \times n},$$

$$1 \leq i \leq m$$

$$1 \leq j \leq n$$

Note that number of rows are written first and then the number of columns.

In the example given above, A is a 3×2 matrix but C is a 2×3 matrix.

Note : In this chapter

- (i) We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
- (ii) We shall consider only those matrices whose elements are real numbers.

WORKED OUT EXAMPLES :

Example 1: Consider the following information regarding the number of girls and boys in three sections A, B, C of a class.

	Girls	Boys
A	30	24
B	26	34
C	28	25

Represent the above information in the form of 3×2 matrix. What does the entry in the third row and second column represent?

Solution : The information is represented in the form of a 3×2 matrix as follows

$$P = \begin{bmatrix} 30 & 24 \\ 26 & 34 \\ 28 & 25 \end{bmatrix}_{3 \times 2}$$

The entry in the third row and second column i.e. 25 represent the number of girls in section C.

Example 2: If a matrix has 12 elements what are the possible orders it can have?

Solution : We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 12 elements we will find ordered pairs of natural numbers, whose product is 12.

Thus all possible ordered pairs are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4,3)

Hence, possible orders are

$$1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4, 4 \times 3$$

Example 3: Construct a 2×3 matrix whose elements are given by $a_{ij} = \frac{1}{2}(i^2 - 2j)$

Solution : In general 2×3 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$\text{Now } a_{ij} = \frac{1}{2}(i^2 - 2j), \quad i = 1, 2; \quad j = 1, 2, 3$$

Therefore,

$$a_{11} = \frac{1}{2}(1^2 - 2) = -\frac{1}{2} \quad a_{12} = \frac{1}{2}(1^2 - 4) = -\frac{3}{2}$$

$$a_{13} = \frac{1}{2}(1^2 - 6) = -\frac{5}{2}$$

$$a_{21} = \frac{1}{2}(2^2 - 2) = 1 \quad a_{22} = \frac{1}{2}(2^2 - 4) = 0$$

$$a_{23} = \frac{1}{2}(2^2 - 6) = -1$$

Hence the required matrix is given by $A = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 1 & 0 & -1 \end{bmatrix}$

TYPES OF MATRICES :

In this section, we shall discuss different types of matrices

(i) **Row Matrix :** A matrix is said to be a row matrix if it has only one row.

$$\text{For example, } A = [5 \quad -6 \quad 2 \quad 7]$$

is a row matrix of order 1×4

In general, $A = [a_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(ii) Column Matrix :

A matrix is said to be a column matrix if it has only one column

$$\text{For example } B = \begin{bmatrix} 4 \\ -7 \\ 3 \\ 6 \end{bmatrix}_{4 \times 1} \text{ is a column matrix.}$$

In general, $B = [b_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(iii) Square Matrix :

A matrix in which the number of rows are equal to the number of columns, it is said to be a square matrix. Thus a $m \times n$ matrix is said to be a square matrix if $m = n$ and is known to be a square matrix of order n .

Note : If $A = [a_{ij}]$ is a square matrix of order n , then the elements $a_{11}, a_{22}, \dots, a_{nn}$ are said to constitute the principal diagonal, of the matrix and are called the diagonal elements of the matrix A . Thus if

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 0 & 4 & 1 \\ 6 & -7 & -4 \end{bmatrix}, \text{ the } 5, 4, -4 \text{ are diagonal elements.}$$

(iv) Null or Zero matrix :

A matrix is said to be zero matrix or null matrix if all its elements are zero.

$$\text{For example, } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[0, 0]$ are all null matrices, we denote a null matrix by 0 (bold 0)

(v) Diagonal Matrix :

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal matrix if all its non diagonal elements are zero, that is $A = [a_{ij}]_{m \times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

$$\text{For example, } A = [7], B = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

are diagonal matrices of order 1, 2, 3 respectively.

(vi) Scalar Matrix :

A square matrix is said to be a scalar matrix if all its non-diagonal elements are equal to zero and diagonal elements are equal.

So, a scalar matrix is a diagonal matrix whose diagonal elements are equal.

That is a square matrix $A = [a_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0 \text{ when } i \neq j$$

$$b_{ij} = k \text{ when } i = j, \text{ for some constant } k.$$

For example

$$A = [5], B = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}, C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are scalar matrix of order 1, 2 and 3 respectively.

(vii) Identity or Unit Matrix :

A square matrix in which elements in the diagonal are all 1 and rest are all zero, is called an identity or a unit matrix. In other words, the square matrix $A = [a_{ij}]_{n \times n}$ is an identity matrix if

$$a_{ij} = 1 \text{ when } i = j$$

$$= 0 \text{ when } i \neq j$$

We denote identity or unit matrix of order n by I_n . When order is clear from the context, we simply write it as I .

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of order 1, 2 and 3 respectively.

Note : (i) An identity matrix is a scalar matrix also. But a scalar matrix is not an identity matrix.

(ii) Every scalar matrix is a diagonal matrix also but the opposite is not true.

(iii) **Triangular Matrix :** A square matrix in which all the elements above or below the Principal diagonal are zero is called a triangular matrix.

An upper triangular matrix is a triangular matrix in which all the elements below the principal diagonal are zero.

A lower triangular matrix is a triangular matrix in which all the elements above the principal diagonal are zero.

For example the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

is an upper triangular matrix and

$$B = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is a lower triangular matrix.

(ix) Singular and Non-Singular Matrix :

A square matrix A is said to be singular if its determinant is equal to zero i.e. $|A| = 0$ and non-singular if $|A| \neq 0$.

For example, $A = \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$ is singular

$$\text{since } |A| = \begin{vmatrix} 3 & -6 \\ 2 & -4 \end{vmatrix} = -12 + 12 = 0$$

and $B = \begin{bmatrix} 7 & 3 \\ -4 & 1 \end{bmatrix}$ is non-singular

$$\text{since } |B| = \begin{vmatrix} 7 & 3 \\ -4 & 1 \end{vmatrix} = -7(-12) = 84 \neq 0$$

3.3.6 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

are said to be equal if

(i) they are of same order.

(ii) each element of A is equal to the corresponding element of B, i.e. $a_{ij} = b_{ij}$ for all i and j

$$\text{For example } \begin{bmatrix} 5 & 3 \\ 1 & 6 \\ -7 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 5 & 3 \\ 1 & 6 \\ -7 & 0 \end{bmatrix}$$

are equal matrices but $\begin{bmatrix} 5 & 3 \\ 1 & 6 \\ -7 & 0 \end{bmatrix}$ and $\begin{bmatrix} -5 & 3 \\ 1 & -6 \\ 7 & 0 \end{bmatrix}$ are not equal matrices.

Symbolically, if two matrices A and B are equal, we write $A = B$

$$\text{If } \begin{bmatrix} x & y \\ p & q \\ a & b \end{bmatrix} = \begin{bmatrix} -3.2 & 0 \\ 5 & \sqrt{6} \\ -8 & 7 \end{bmatrix}$$

$$\begin{aligned} \text{then } x &= -3.2, & y &= 0 \\ p &= 5, & q &= \sqrt{6} \\ a &= -8, & b &= 7 \end{aligned}$$

Example 4:

$$\text{If } \begin{bmatrix} x+3 & 2z-7 & 3y+1 \\ -8 & a-2 & 0 \\ b+4 & -17 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -11 \\ -8 & 5 & 2c-4 \\ 2b+5 & -17 & 0 \end{bmatrix}$$

Find the values of a, b, c, x, y and z.

Solution : As the given matrices are equal, their corresponding elements must be equal.
Comparing the corresponding the elements we get

$$\begin{aligned} x+3 &= 0, & 2z-7 &= 3, & 3y+1 &= -11 \\ a-2 &= 5, & 2c-4 &= 0, & 2b+5 &= b+4 \end{aligned}$$

Simplifying we get

$$\begin{aligned} x &= -3, & z &= 5, & y &= -4 \\ a &= 7, & c &= 2, & b &= -1 \end{aligned}$$

OPERATIONS WITH MATRICES :

Now, we are going to introduce certain operations on matrices namely addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.3.7 ADDITION OF MATRICES

Consider the following examples : There are two suppliers A and B who supplies brick, sand and stone. A has supplied 5 trucks of brick, 3 trucks of sand and 2 trucks of stone to Mr. Choudhury and 3 trucks of brick, 4 trucks of sand and 3 trucks of stone to Mr. Ahmed. Again Firm B has supplied 3 trucks of brick, 4 trucks of sand and 3 trucks of stone to Mr. Choudhury and 2 trucks of brick, 2 trucks of sand and 1 truck of stone to Mr. Ahmed. The above informations regarding supplies to Mr. Choudhury and Mr. Ahmed may be represented as matrices given below.

	Mr. Choudhury		Mr. Ahmed	
	FirmA	FirmB	FirmA	FirmB
Brick	5	3	3	2
Sand	3	4	4	2
Stone	2	3	3	1

If we want to know the total supplies made for each category by firm A and B, then

Brick :	Firm A (5+3);	Firm B (3+2)
Sand :	Firm A (3+4);	Firm B (4+2)
Stone :	Firm A (2+3);	Firm B (3+1)

This may be represented in the matrix form as :

	FirmA	FirmB
Brick	5 + 3	3 + 2
Sand	3 + 4	4 + 2
Stone	2 + 3	3 + 1

This new matrix is the sum of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Moreover, the two matrices must be of same order.

In general, if $A=[a_{ij}]$ and $B=[b_{ij}]$ are two matrices of the same order, say $m \times n$, then their sum is denoted by $A+B$ where $A+B = [a_{ij}+b_{ij}]$ is also a matrix of order $m \times n$.

Example 5: Given

$$A = \begin{bmatrix} -5 & 7 \\ 4 & 2 \\ 6 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -1 \\ 2 & 5 \\ 3 & 1 \end{bmatrix}$$

Find A+B

Solution : As A and B are both of the same order 3×2 , therefore A+B is defined and

$$A + B = \begin{bmatrix} -5+6 & 7+(-1) \\ 4+2 & 2+5 \\ 6+3 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 6 & 7 \\ 9 & 1 \end{bmatrix}$$

Note : It is to be noted that if A and B are not of the same order, then A+B is not defined. For example, if

$$A = \begin{bmatrix} 5 & 6 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & -1 \\ 6 & 2 & 3 \end{bmatrix}$$

then A+B is not defined.

3.3.8 MULTIPLICATION OF A MATRIX BY A SCALAR

Now suppose that if Mr. Ahmed has doubled his order in all categories of each firm then the new supplies made to Mr. Ahmed are as follows :

	FirmA	FirmB
Brick	2×3	2×2
Sand	2×4	2×2
Stone	2×3	2×1

This may be represented in the matrix form as

$$\begin{bmatrix} 6 & 4 \\ 8 & 4 \\ 6 & 2 \end{bmatrix}$$

We observe that this new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, if $A = [a_{ij}]$ is a matrix of order $m \times n$ and k is a real number, then kA is another

matrix of order $m \times n$ which is obtained by multiplying each element of A by the real number k . This is called scalar multiplication of matrix A by the scalar k .

Example 6: If $A = \begin{bmatrix} 4 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix}$

Find $3A$.

Solution : Given $A = \begin{bmatrix} 4 & 3 & 9 \\ 2 & 0 & 1 \end{bmatrix}$

$$\therefore 3A = \begin{bmatrix} 12 & 9 & 27 \\ 6 & 0 & 3 \end{bmatrix}$$

3.3.9 NEGATIVE OF A MATRIX

The negative of a matrix A is denoted by $-A$. We define $-A = (-1)A$

For example let $A = \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix}$

$$\begin{aligned} \text{Then } -A &= (-1)A = (-1) \begin{bmatrix} 5 & 2 \\ -3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & -2 \\ 3 & -4 \end{bmatrix} \end{aligned}$$

3.3.10 DIFFERENCE OF MATRICES :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order $m \times n$, then their difference is denoted by $A - B$ where

$A - B = [a_{ij} - b_{ij}]$ is also a the matrix of order $m \times n$.

In other words, $A - B$

$$= A + (-1)B$$

that is sum of matrix A and the matrix $-B$.

Example 6: If $A = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 6 & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} -3 & 6 & -4 \\ 5 & 2 & 3 \end{bmatrix}, \text{ Find } 3A - 2B.$$

Solution : Given $A = \begin{bmatrix} 5 & 4 & 2 \\ 1 & 6 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} -3 & 6 & -4 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\therefore 3A-2B$$

$$= 3 \begin{bmatrix} 5 & 4 & 2 \\ 1 & 6 & 0 \end{bmatrix} - 2 \begin{bmatrix} -3 & 6 & -4 \\ 5 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 12 & 6 \\ 3 & 18 & 0 \end{bmatrix} - \begin{bmatrix} -6 & 12 & -8 \\ 10 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 - (-6) & 12 - 12 & 6 - (-8) \\ 3 - 10 & 18 - 4 & 0 - 6 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 14 \\ -7 & 14 & -6 \end{bmatrix}$$

Example 7 : If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -3 \\ 5 & 1 \\ -6 & 2 \end{bmatrix}$

then find matrix X such that $2A+3x-4B = 0$

Solution : Given $2A+3x-4B = 0$

$$\Rightarrow 3x = 4B-2A$$

$$\Rightarrow x = \frac{1}{3} (4B-2A) \dots\dots\dots (1)$$

Now $4B-2A$

$$= \begin{bmatrix} 12 & -12 \\ 20 & 4 \\ -24 & 8 \end{bmatrix} - \begin{bmatrix} 16 & 0 \\ 8 & -4 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -12 \\ 12 & 8 \\ -34 & 2 \end{bmatrix}$$

\therefore From, (1)

$$x = \frac{1}{3} (4B-2A)$$

$$= \frac{1}{3} \begin{bmatrix} -4 & -12 \\ 12 & 8 \\ -34 & 2 \end{bmatrix} = \begin{bmatrix} -4/3 & -4 \\ 4 & 8/3 \\ -34/3 & 2/3 \end{bmatrix}$$

Example 8 : An industrialist produces 3 products A, B and C which he sells in two cities Guwahati and Delhi. The annual sales for 2008 and 2009 are as follows.

2008			
	A	B	C
Guwahati	2000	8000	10,000
Delhi	20,000	4000	6000

2009			
	A	B	C
Guwahati	1500	6500	8000
Delhi	14,000	1500	6000

- (i) Find the combined sales in 2008 and 2009 for each type.
- (ii) Find the decrease in sales from 2008 to 2009.
- (iii) If the profit made in 2008 is 30% of the gross sale compute the total profit made in the two cities in 2008 for each type.

Solution : Let P and Q are the matrix which denotes the sales in 2008 and 2009.

$$\therefore P = \begin{bmatrix} \text{A} & \text{B} & \text{C} \\ 2000 & 8000 & 10,000 \\ 20,000 & 4000 & 6000 \end{bmatrix} \begin{matrix} \text{Guwahati} \\ \text{Delhi} \end{matrix}$$

$$\therefore Q = \begin{bmatrix} 1500 & 6500 & 8000 \\ 14,000 & 1500 & 6000 \end{bmatrix} \begin{matrix} \text{Guwahati} \\ \text{Delhi} \end{matrix}$$

- (i) \therefore Combined sale in 2008 and 2009

$$= P + Q = \begin{bmatrix} \text{A} & \text{B} & \text{C} \\ 3500 & 14,500 & 18,000 \\ 34,000 & 5,5000 & 12,000 \end{bmatrix} \begin{matrix} \text{Guwahati} \\ \text{Delhi} \end{matrix}$$

(ii) Change in the sales from 2008 to 2009 is given by

$$P - Q = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 500 & 1500 & 2000 \\ 6000 & 2500 & 0 \end{bmatrix} & \begin{matrix} \text{Guwahati} \\ \text{Delhi} \end{matrix} \end{matrix}$$

(iii) Gross profit made in two cities for each type of products in two cities in 2008 = 30% of P

$$= 0.3 \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 2000 & 8000 & 10,000 \\ 20,000 & 4000 & 6000 \end{bmatrix} & \end{matrix}$$

$$= \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 600 & 2400 & 3000 \\ 6000 & 1200 & 1800 \end{bmatrix} & \begin{matrix} \text{Guwahati} \\ \text{Delhi} \end{matrix} \end{matrix}$$

\therefore In 2008, total gross profit made for products A, B and C respectively in Guwahati and Delhi are– Rs. 600, Rs. 2400, Rs. 3000 and Rs. 6000, Rs. 1200, Rs. 1800.

3.1.12 MULTIPLICATION OF MATRICES

Suppose Reba and Pampy are two friends. Reba wants to buy 3 pens and 8 exercise books and Pampy wants to buy 6 pens and 12 exercise books. They both go to shop to enquire about the price which are as follows–

Pen – Rs. 6 each

Exercise Book – Rs. 55 each

How much does each need to spend?

Clearly, Reba needs Rs. $(6 \times 3 + 55 \times 8) = \text{Rs. } 458$ and Pampy needs Rs. $(6 \times 6 + 55 \times 12) = \text{Rs. } 696$. In terms of matrix representation, we can write the above information as follows :

Requirement	Price Per Unit (Rs.)	Money needed (Rs.)								
<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">Pen</td> <td style="padding-right: 10px;">Exercise Book</td> <td></td> </tr> <tr> <td style="padding-right: 10px;">$\begin{bmatrix} 3 \\ 6 \end{bmatrix}$</td> <td style="padding-right: 10px;">$\begin{bmatrix} 8 \\ 12 \end{bmatrix}$</td> <td style="padding-right: 10px;">$\begin{matrix} \text{Reba} \\ \text{Pampy} \end{matrix}$</td> </tr> </table>	Pen	Exercise Book		$\begin{bmatrix} 3 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 12 \end{bmatrix}$	$\begin{matrix} \text{Reba} \\ \text{Pampy} \end{matrix}$	<table style="display: inline-table; border: none;"> <tr> <td style="padding-right: 10px;">$\begin{bmatrix} 6 \\ 55 \end{bmatrix}$</td> <td style="padding-right: 10px;">$\begin{matrix} \text{Pen} \\ \text{Exercise Book} \end{matrix}$</td> </tr> </table>	$\begin{bmatrix} 6 \\ 55 \end{bmatrix}$	$\begin{matrix} \text{Pen} \\ \text{Exercise Book} \end{matrix}$	$\begin{bmatrix} 3 \times 6 + 8 \times 55 \\ 6 \times 6 + 12 \times 55 \end{bmatrix} = \begin{bmatrix} 458 \\ 696 \end{bmatrix}$
Pen	Exercise Book									
$\begin{bmatrix} 3 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 12 \end{bmatrix}$	$\begin{matrix} \text{Reba} \\ \text{Pampy} \end{matrix}$								
$\begin{bmatrix} 6 \\ 55 \end{bmatrix}$	$\begin{matrix} \text{Pen} \\ \text{Exercise Book} \end{matrix}$									

Suppose that they enquire about the rates from another shop, where the prices are –

Pen – Rs. 5 each

Exercise Book – Rs. 50 each

So, the above information for the second shop may be represented in matrix form as follows :

$$\begin{array}{ccc}
 \text{Requirement} & \text{Price Per Unit(Rs.)} & \text{Money needed (Rs.)} \\
 \begin{bmatrix} 3 & 8 \\ 6 & 12 \end{bmatrix} & \begin{bmatrix} 5 \\ 50 \end{bmatrix} & \begin{bmatrix} 3 \times 5 + 8 \times 50 \\ 6 \times 5 + 12 \times 50 \end{bmatrix} = \begin{bmatrix} 415 \\ 630 \end{bmatrix}
 \end{array}$$

Now the information in both the cases may be combined and expressed in terms of matrices as follows :

$$\begin{array}{ccc}
 \text{Requirement} & \text{Price Per Unit(Rs.)} & \text{Money needed (Rs.)} \\
 \begin{bmatrix} 3 & 8 \\ 6 & 12 \end{bmatrix} & \begin{bmatrix} 6 & 5 \\ 55 & 50 \end{bmatrix} & = \begin{bmatrix} 3 \times 6 + 8 \times 55 & 3 \times 5 + 8 \times 50 \\ 6 \times 6 + 12 \times 55 & 6 \times 5 + 12 \times 50 \end{bmatrix} = \begin{bmatrix} 458 & 415 \\ 696 & 630 \end{bmatrix}
 \end{array}$$

The above is an example of multiplication of matrices. We observe that for multiplication of two matrices, A and B, the number of columns in A should be equal to the number of rows in B. Moreover, for getting the elements of the product matrix, we take the rows of A and columns of B, multiply them elementwise and take the sum.

Thus the product of any two matrices A and B are defined if No. of columns of A (1st matrix)

= No. of rows of B (2nd matrix)

Let A = [a_{ij}] be a m×n matrix and B=[b_{jk}] be a n×p matrix.

Then their product AB=C is a matrix of order m×p i.e. C=[C_{ik}].

Suppose we want to get the element of the ith row and kth column of AB.

We take ith row of A=[a_{i1} a_{i2} a_{in}] and kth column of B = $\begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{nk} \end{bmatrix}$

Multiplying elementwise and adding the ith row and kth column of C we get,

$$\begin{aligned}
 C_{ik} &= a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \dots + a_{in} b_{nk} \\
 &= \sum_{j=1}^n a_{ij} b_{jk}
 \end{aligned}$$

Thus if $A = [a_{ij}]_{m \times n}$
 $B = [b_{jk}]_{n \times p}$ then their product
 $AB = [C_{ik}]_{m \times p}$

where $C_{ik} = \sum_{j=1}^n a_{ij} b_{jk}$

For example, if $A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} 1 & 6 & 5 \\ 4 & 7 & 2 \end{bmatrix}_{2 \times 3}$

So here AB is defined and it is 2×3 matrix.

But BA is not defined as Number of columns of B \neq Number of Rows of A

Now $AB = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 6 & 5 \\ 4 & 7 & 2 \end{bmatrix}$ is 2×3 matrix

in which each entry is the sum of products across some row of A with the corresponding entries down some columns of D. These six entries are—

$$\begin{aligned} \text{Entry in the 1st row} &= \begin{pmatrix} \boxed{2} & \boxed{3} & \boxed{1} & \boxed{6} & \boxed{5} \\ \boxed{5} & \boxed{6} & \boxed{4} & \boxed{7} & \boxed{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 1 + 3 \times 4 & 2 \times 6 + 3 \times 7 & 2 \times 5 + 3 \times 2 \\ - & - & - \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Entry in the 2nd row} &= \begin{pmatrix} \boxed{2} & \boxed{3} & \boxed{1} & \boxed{6} & \boxed{5} \\ \boxed{5} & \boxed{6} & \boxed{4} & \boxed{7} & \boxed{2} \end{pmatrix} \\ &= \begin{pmatrix} - & - & - \\ 5 \times 1 + 6 \times 4 & 2 \times 6 + 3 \times 7 & 5 \times 5 + 6 \times 2 \end{pmatrix} \end{aligned}$$

$$= \begin{bmatrix} 18 & 33 & 16 \\ 29 & 72 & 37 \end{bmatrix}$$

Note : (i) In multiplication of two matrices A and B, if AB is defined, then BA need not be defined.

if A and B are respectively. $m \times n$ and $k \times l$ matrices, then both AB and BA are defined if and only if $n=k$ and $m=l$.

In particular, if A and B are both square matrices of same order, then both AB and BA are defined.

(ii) It is also seen that even if both AB and BA are defined, it is not necessary that $AB=BA$.

Example 9: If $A = \begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}$

find AB and BA. Are they equal?

Solution: Given $A = \begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}$

Here AB and BA both are defined and each is a 2×2 matrix.

$$\begin{aligned} \text{Now, } AB &= \begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 15+6 & 20-36 \\ -3+2 & -4-12 \end{bmatrix} = \begin{bmatrix} 21 & -16 \\ -1 & -16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } BA &= \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 15-4 & 18+8 \\ 5+6 & 6-12 \end{bmatrix} = \begin{bmatrix} 11 & 26 \\ 11 & -6 \end{bmatrix} \end{aligned}$$

Here $AB \neq BA$.

Note : 1. Here AB and BA are of same order but $AB \neq BA$.

2. This does not mean that $AB \neq BA$ for every pair of matrices A, B for which AB and BA are defined.

For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{aligned} \text{Then} \quad AB &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+0 & 0+0 \\ 0+0 & 0+8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad BA &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3+0 & 0+0 \\ 0+0 & 0+8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned}$$

Here $AB = BA$

Zero matrix as the product of two non zero matrices :

We know that, for real numbers a and b , if $ab=0$ then either $a=0$ or $b=0$. But this is not true for matrices.

$$\text{For example, Let } A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \quad B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{Then} \quad AB &= \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Thus if product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.1.13 PROPERTIES OF MATRIX MULTIPLICATION :

1. The associative Law : For any three matrices A , B and C .
 $(AB)C = A(BC)$ when both sides of the equality are defined.
2. The distributive Law : For three matrices A , B and C .
 - (i) $(A+B)C = AC+BC$
 - (ii) $A(B+C) = AB+AC$
 whenever both sides of equality are defined.
3. The existence of multiplicative identity : For every square matrix A , there exists an identity matrix of the same order I of the order such that $AI = IA = A$

3.1.14 TRANSPOSE OF A MATRIX

Let A be a $m \times n$ matrix. The matrix obtained from A by interchanging its rows into columns and columns into rows is called transpose of A and is denoted by A' or A^T which is a $n \times m$ matrix.

$$\text{If } A = (a_{ij})_{m \times n}$$

$$\text{Then } A' = (b_{ij})_{n \times m}$$

$$\text{where } b_{ij} = a_{ji}$$

$$\text{For example, if } A = \begin{pmatrix} 2 & -6 & 4 \\ 0 & 1 & 7 \end{pmatrix}_{2 \times 3}$$

$$\text{Then } A' = \begin{pmatrix} 2 & 0 \\ -6 & 1 \\ 4 & 7 \end{pmatrix}_{3 \times 2}$$

Properties of Transpose of a matrix :

- (i) Transpose of transpose of a matrix A is again A ie $(A')' = A$
- (ii) $(A+B)' = A'+B'$
- (iii) $(AB)' = B'A'$
- (iv) If k is any scalar $(kA)' = kA'$

Orthogonal Matrix :

A square matrix A is said to be orthogonal if $AA' = A'A = I$ where I is the unit matrix of same order that of A .

3.1.15 SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A = A'$.

Thus if $A = (a_{ij})_{n \times n}$ then A will be symmetric if $a_{ij} = a_{ji}$ for all i, j

$$\text{For example, } A = \begin{bmatrix} a & h^{a_{12}} & g^{a_{13}} \\ h & b & f^{a_{23}} \\ a^{a_{21}} & g & f_{a_{32}} & c \end{bmatrix}$$

is a symmetric matrix.

Example 10: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix}$

$$B = \begin{pmatrix} 3 & -4 \\ 1 & 6 \\ -2 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 5 & -1 \\ 3 & 0 \end{pmatrix}$$

Prove that $A(BC) = (AB)C$

Solution : Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix}$

$$B = \begin{pmatrix} 3 & -4 \\ 1 & 6 \\ -2 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 5 & -1 \\ 3 & 0 \end{pmatrix}$$

Now $BC = \begin{pmatrix} 3 & -4 \\ 1 & 6 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 3 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 15-12 & -3+0 \\ 5+18 & -1+0 \\ -10+6 & 2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 \\ 23 & -1 \\ -4 & 2 \end{pmatrix}$$

\therefore L.H.S. = $A(BC)$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 23 & -1 \\ -4 & 2 \end{pmatrix}$$

$$\begin{aligned}
 &= \begin{pmatrix} 3+46-12 & -3-2+6 \\ 12+0-20 & -12+0+10 \end{pmatrix} \\
 &= \begin{pmatrix} 37 & 1 \\ -8 & -2 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } AB &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & 6 \\ -2 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 3+2-6 & -4+12+6 \\ 12+0-10 & -16+0+10 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 14 \\ 2 & -6 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{R.H.S.} = (AB)C &= \begin{pmatrix} -1 & -14 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 3 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -5+42 & 1+0 \\ 10-18 & -2+0 \end{pmatrix} \\
 &= \begin{pmatrix} 37 & 1 \\ -8 & -2 \end{pmatrix}
 \end{aligned}$$

$\therefore A(BC) = (AB)C$ (Hence Proved)

Example 11: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

Prove that $A^2 = 0$

Solution :

[Note that for a matrix A, A^2 does not mean squaring each element of the matrix A. A^2 means A.A]

$$\begin{aligned}
 \text{Here } A^2 &= A.A \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}
 \end{aligned}$$

$$= \begin{pmatrix} 1+2-3 & 2+4-6 & 3+6-9 \\ 1+2-3 & 2+4-6 & 3+6-9 \\ -1-2+3 & -2-4+6 & -3-6+9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

$\therefore A^2 = 0$ (Hence Proved)

Example 12: If $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$

Show that $A^3 - 2A^2 + 4A - 18I = 0$

Solution : Given $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$

$$\therefore A^2 = A.A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2+2 & 2+0-1 & 1+6+1 \\ -1+0+6 & -2+0-3 & -1+0+3 \\ 2+1+2 & 4+0-1 & 2-3+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{pmatrix}$$

$$A^3 = A^2.A = \begin{pmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1+16 & 2+0-8 & 1+3+8 \\ 5+5+4 & 10+0-2 & 5-15+2 \\ 5-3+0 & 10+0+0 & 5+9+0 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{pmatrix}$$

$$\therefore \text{L.H.S.} = A^3 - 2A^2 + 4A - 18I$$

$$= \begin{pmatrix} 16 & -6 & 12 \\ 14 & 8 & -8 \\ 2 & 10 & 14 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 & 8 \\ 5 & -5 & 2 \\ 5 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} - 18 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16-2+4-18 & -6-2+8-0 & 12-16+4-0 \\ 14-10-4-0 & 8+10+0-18 & -8-4+12-0 \\ 2-10+8-0 & 10-6-4-0 & 14-0+4-18 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0 = \text{R.H.S.}$$

(Hence Shown)

Example 13: There are two families X and Y. In family X, there are 3 men, 5 women and 6 children and in family Y, there are 2 men, 2 women and 3 children. The recommended daily requirement of protein and carbohydrates for men, women and children are as follows :

Protein : Men : 50gm; Women : 40gm; Children : 35gm.

Carbohydrates

Men : 60gm; Women: 45gm; Children : 25gm.

Calculate the total requirement of Protein and Carbohydrates for each of the two families by matrix method.

Solution : Let F = Family Matrix

$$= \begin{matrix} & \begin{matrix} X & Y \end{matrix} \\ \begin{matrix} 3 & 2 \\ 5 & 2 \\ 6 & 3 \end{matrix} & \begin{matrix} \text{Men} \\ \text{Woman} \\ \text{Children} \end{matrix} \end{matrix} \\ 3 \times 2$$

and R = Requirement Matrix

$$= \begin{array}{ccc} \text{Men} & \text{Woman} & \text{Children} \\ \left[\begin{array}{ccc} 50 & 40 & 35 \\ 60 & 45 & 25 \end{array} \right] \begin{array}{l} \text{Protein} \\ \text{Carbohydrate} \end{array} \end{array}$$

\therefore Total requirement = RF

$$= \begin{array}{ccc} \left[\begin{array}{ccc} 50 & 40 & 35 \\ 60 & 45 & 25 \end{array} \right] \begin{array}{cc} \left[\begin{array}{cc} 3 & 2 \\ 5 & 2 \\ 6 & 3 \end{array} \right] \\ \\ \left[\begin{array}{cc} 150 + 200 + 210 & 100 + 80 + 105 \\ 180 + 225 + 150 & 120 + 90 + 75 \end{array} \right] \end{array}$$

$$= \begin{array}{cc} \begin{array}{cc} \text{X} & \text{Y} \\ \left[\begin{array}{cc} 560 & 285 \\ 555 & 285 \end{array} \right] \begin{array}{l} \text{Protein} \\ \text{Carbohydrate} \end{array} \\ 3 \times 2 \end{array} \end{array}$$

\therefore Total requirement for family X is

Protein : 560 gm

Carbohydrate : 555 gm

Total requirement for family Y is

Protein : 285 gm

Carbohydrate : 285 gm

Note : Here both FR and RF are defined. Where FR is a 3×3 and RF is a 2×2 matrix.

But we require total requirement of protein and carbohydrates for family X and Y. So, we have computed RF which is a matrix having $2 \times 2 = 4$ elements

SUMMARY

- ★ A matrix is an ordered rectangular arrangement of numbers in rows and columns.
- ★ A matrix having m rows and n columns is called a matrix of order $m \times n$.
- ★ $[a_{ij}]_{m \times 1}$ is a column matrix.
- ★ $[a_{ij}]_{1 \times n}$ is a row matrix.
- ★ An $m \times n$ matrix is a square matrix if $m=n$.
- ★ $A=[a_{ij}]_{m \times n}$ is a diagonal matrix if $a_{ij}=0$ when $i \neq j$.
- ★ $A=[a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij}=0$ when $i \neq j$ and $a_{ij}=k$ (k is some constant) when $i=j$.
- ★ $A=[a_{ij}]_{n \times n}$ is an identity matrix if $a_{ij}=0$ when $i \neq j$ and $a_{ij}=1$ when $i=j$.
- ★ $A=[a_{ij}]_{m \times n}$ is a zero or null matrix if $a_{ij}=0$ for all i and j .
- ★ Two matrices $A=[a_{ij}]_{m \times n}$ and $B=[b_{ij}]_{m \times n}$ are said to be equal if $a_{ij}=b_{ij}$ for all i and j .
- ★ $kA=k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- ★ $A+B = B+A$
- ★ $A-B = A+(-1)B$
- ★ If $A = [a_{ij}]_{m \times n}$ and $B=[b_{ij}]_{n \times p}$, then $AB=C=[C_{ik}]_{m \times p}$ where $C_{ik} = \sum_{j=1}^n a_{ij} b_{jk}$
- ★ In general $AB \neq BA$
- ★ $A(BC) = (AB)C$
- ★ $A(B+C) = AB+AC$
- ★ $(A+B)C = AC+BC$
- ★ If $A=[a_{ij}]_{m \times n}$ then A' or $A^T = (b_{ij})_{n \times m}$ where $b_{ij} = a_{ji}$
- ★ A is a symmetric matrix, $(A')' = A$
- ★ $(A+B)' = A'+B'$
- ★ $(AB)' = B'A'$
- ★ $(kA)' = kA'$ where k is any real number.
- ★ A square matrix A is said to be orthogonal if $AA' = A'A=I$

EXERCISE 3.3

1. If a matrix has 12 elements, what are the possible order it may have? What will be the possible order if it has 5 elements?
2. Write down the indicated elements of the following matrix.

(i) a_{13} , a_{22} and a_{32} in

$$A = \begin{bmatrix} 5 & -6 & 0 \\ 4 & 3 & 7 \\ -1 & 2 & -3 \end{bmatrix}$$

(ii) b_{14} , b_{23} and b_{34} in

$$B = \begin{bmatrix} 2 & -5 & 6 & 8 \\ 0 & 4 & -2 & 1 \\ -4 & 9 & 10 & 7 \end{bmatrix}$$

3. (i) Construct a 2×3 matrix where

$$a_{ij} = \frac{2j-j}{j^2}$$

(ii) Construct a 4×2 matrix where

$$a_{ij} = \frac{j^2}{2}$$

4. Find the value of x , y , z , w if

$$\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 7 & -4 & 3 \\ -6 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 8 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

Find (i) $3A+4B$

(ii) $2A-3B$

6. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

$$\text{and } C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

- Compute
- (i) $A+B$
 - (ii) $3C-A$
 - (iii) AB
 - (iv) $C(A+B)$

7. Find the matrix X such that $3A-2B+X=0$ where $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

8. Find x and y if

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

9. Find X and Y if

(i) $X+Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X-Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) Find X if $Y = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

$$2x+Y = \begin{bmatrix} 1 & -3 \\ 7 & 0 \end{bmatrix}$$

10. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ and

$(A+B)^2 = A^2+B^2$ find the values of x and y

11. If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Prove that $(aI+bA)^3 = a^3 I + 3a^2bA$

12. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$ $C = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}$

- verify that
- (i) $A.(B.C) = (A.B).C$
 - (ii) $(A+B).C = A.C.+B.C.$

13. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that it satisfies the equation $A^2 - (a+d)A + (ad-bc)I = 0$

14. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

find $A^2 - 5A + 6I$

15. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Prove that $A^3 - 6A^2 + 7A + 2I = 0$

16. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

find k , such that $A^2 = kA - 2I$.

17. Show that the matrix $\frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

is orthogonal.

18. If $A = \begin{pmatrix} 5 & 3 \\ 0 & -4 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -6 \\ -1 & 7 \end{pmatrix}$

Show that $(AB)' = B'A'$

19. Three firms X, Y and Z supplied 30, 25 and 15 trucks of brick and 12, 3 and 7 trucks of sand respectively to Mr. Hazarika. If the cost of brick and sand per truck are respectively, Rs.5400 and Rs.3000, find the total amount paid by the Mr. Hazarika to each firm by using matrix method.

20. In a certain district there are 15 branch office and 45 customer service centre of a mobile company. Each branch office has 1 incharge, 4 office assistant, 2 computer operator and 2 peons. On the other hand each customer service centre has 1 office assistant, 3 computer operator and 1 peon. The salary of each of them are as follows :

Peon : Rs.2500; **Computer Operator :** Rs.7200; **Office Assistant :** Rs.9500
and **Incharge :** Rs.16,500.

Using matrix operation find (i) Total monthly salary bill of each branch office and Customer Service Centre.

(ii) The total monthly salary bill of the whole district.

ANSWERS 3.3

1. $1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4, 4 \times 3, 5 \times 1, 1 \times 5.$

2. (i) $a_{13}=0, a_{22}=3, a_{32}=2$

(ii) $b_{14}=8, b_{23}=-2, b_{34}=7.$

3. (i) $A = \begin{bmatrix} 1 & 0 & -1 \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

(ii) $A = \begin{bmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 \\ \frac{1}{2} & 2 \end{bmatrix}$

4. $x = 2, y = -1, z = 1, w = 4$

5. (i) $\begin{bmatrix} 9 & 20 & 1 \\ -18 & -6 & 13 \end{bmatrix}$

(ii) $\begin{bmatrix} 23 & -32 & 12 \\ -12 & 13 & -14 \end{bmatrix}$

6. (i) $\begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 11 & 1 & 9 \\ -5 & 9 & 4 \\ 2 & -5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 31 & 4 & 11 \\ 33 & 4 & 29 \\ -5 & -6 & -3 \end{bmatrix}$

7. $x = \begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$

8. $x = 3, y = -4$

9. (i) $x = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(ii) $x = \begin{bmatrix} -1 & -5/2 \\ 1 & -2 \end{bmatrix}$

10. $x = 1, y = 4$

14. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

16. $k = 1.$

19. Firm X = Rs. 1,98,000

Firm Y = Rs. 1,44,000

Firm Z = Rs. 1,02,000

20. (i) Branch Office = Rs. 73,900

Customer Service Centre = Rs. 33,600

(ii) Rs. 26,20,500

UNIT-4**STATISTICAL AVERAGES****OBJECTIVES :**

This chapter will help you to understand–

- ➔ The meaning, purpose and characteristics of an average.
- ➔ The different types of averages, their properties, merits, demerits and uses.
- ➔ The calculation of averages for individual series, ungrouped and grouped frequency distributions.
- ➔ The meaning and calculation of partitions values.
- ➔ The graphical representation of Median and Mode.

4.1 INTRODUCTION :

The arithmetical measures which effectively condense the data to a single value are called Averages. They represent the entire mass of unwieldy data thereby reducing the complexity of the raw data. Besides condensing the data, averages provide a more precise description of the distribution and further helps in comparing two or more similar distributions. Since an average represents the entire data, its value lies between the two extreme observations. i.e., the largest and the smallest observations. There is a general tendency of the value of most of the frequency distributions to concentrate in the middle part of the distributions. For this reasons, an average is referred to as a **measure of central tendency**. Averages or Measures of central tendency are also known as **measures of locations** as they enable us to locate the position of a distributions.

The following definitions of an average by different statisticians will help us to have a clear idea about it.

Clark defined average as – “Average is an attempt to find one single figure to describe whole of figures.”

According to Croxton and Cowden – “An average value is a single value within the range of the data that is used to represent all the values in the series. Since an average is somewhere within the range of the data, it is sometimes called a measure of central tendency.” The definition given by Croxton and Cowden is adequate.

4.2 PURPOSE OF AN AVERAGE :

There are two main purposes of an average. They are–

- (i) **To get a single value to represent the entire group of values.** An average

condenses the mass of data into a single value which becomes easier to understand and remember.

- (ii) **To facilitate comparison.** An average reduces the mass of data to a single figure and thus enables us to make a comparative study of different distributions.

4.3 CHARACTERISTICS OF A GOOD AVERAGE :

A good average or measure of central tendency must pass the following essential qualities or characteristics:

- (i) **It should be rigidly defined** i.e., the definition must be clear so that its interpretation does not differ from person to person.
- (ii) **It should be easy to understand and calculate.**
- (iii) **It should be based on all the observations.** An average should take into consideration each and every observation of a series.
- (iv) **It should be capable of further algebraic treatment.** An average should be such that it can be used for further statistical computations to enhance its utility.
- (v) **It should have sampling stability** i.e., it should be affected as little as possible by fluctuations of sampling.
- (vi) **It should not be unduly affected by extreme observations** i.e., one or two very large or very small values must not effect the average by unduly increasing or decreasing its values.

4.4 TYPES OF AVERAGES :

The following are the important measures of central tendency or averages which are commonly used in practice.

- (i) Arithmetic Mean (A.M.) or Mean
- (ii) Geometric Mean (G.M.)
- (iii) Harmonic Mean (H.M.)
- (iv) Median
- (v) Mode

Remark 1 : Unless specially mentioned mean usually refers to A.M.

Remark 2 : Unit of an average is same as that of the distribution from which it is calculated.

E.g, If unit of a distributions is in 'cm' then the average calculated from that distribution will also be in 'cm'.

Remark 3 : The symbol Σ is the letter capital sigma of the Greek alphabet and is used in Mathematics to denote the sum of values.

For example :

$$(i) x_1 + x_2 + \dots + x_n = \sum_{\ell=1}^n x_i \text{ or } \sum x.$$

$$(ii) x_4 + x_5 + x_6 + x_7 = \sum_{\ell=4}^7 x_i$$

$$(iii) f_1 x_1 + f_2 x_2 + \dots + f_n x_n = \sum_{\ell=1}^n f^i x^i \text{ or } \sum fx.$$

$$(iv) \sum_{\ell=1}^n a = n_a \text{ (where a is a constant)}$$

$$(v) \sum_{\ell=1}^n 4 = 4n \text{ \& } \sum_{\ell=1}^4 3 = 3 + 3 + 3 + 3 = 4 \times 3 = 12$$

$$(vi) \sum_{\ell=1}^3 (x_i + 2) = (x_1 + 2) + (x_2 + 2) + (x_3 + 2)$$

$$= (x_1 + x_2 + x_3) + (2 + 2 + 2)$$

$$= \sum_{\ell=1}^3 x_i + 3 \cdot 2.$$

$$\text{Ily, } \sum_{\ell=1}^n a(x_i + a) = \sum_{\ell=1}^n x_i + n_a.$$

4.5 ARITHMETIC MEAN :

Arithmetic mean of a given set of observations is calculated by adding the values of all the observations and dividing the total by the number of observations.

If x_1, x_2, \dots, x_n are ‘n’ observations, then their AM, usually denoted by \bar{x} is given by

$$AM = \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{\ell=1}^n x_i}{n}$$

where $\sum_{\ell=1}^n x_i$ is the sum of the ‘n’ observations.

Example 1 :

Find the AM of 5, 2, 7, 3, 4.

Solution : $AM = \bar{x} = \frac{5+2+7+3+4}{5} = \frac{21}{5} = 4.5$

4.5.1 CALCULATION OF ARITHMETIC MEAN (AM)**(A) In case of frequency distributions :**

If 'n' variates x_1, x_2, \dots, x_n occurs f_1, f_2, \dots, f_n times respectively i.e., if f_1, f_2, \dots, f_n are the respective frequencies of x_1, x_2, \dots, x_n which can be expressed as

$$\begin{array}{c|cccc} x & x_1 & x_2 & x_3 & \dots & x_n \\ f & f_1 & f_2 & f_3 & \dots & f_n \end{array}$$

then the AM \bar{x} is given by

$$AM(\bar{x}) = \frac{(x_1 + x_1 + \dots + f_1 \text{ times}) + (x_2 + x_2 + \dots + f_2 \text{ times}) + (x_n + x_n + \dots + f_n \text{ times})}{f_1 + f_2 + \dots + f_n}$$

$$= \frac{f_1 x_1 + f_2 x_2 + \dots + f_n x_n}{f_1 + f_2 + \dots + f_n}$$

$$= \frac{\sum_{i=1}^n f_i x_i}{\sum_{i=1}^n f_i}$$

$$= \frac{\sum_{i=1}^n f_i x_i}{N}$$

Where $\sum_{i=1}^n f_i = N$, the total frequency.

Example 2 :

Compute the mean from the following data—

Value (x) :	10	20	30	40	50
Frequency (f) :	7	12	16	10	5

Solution : Let us prepare the following table–

x	f	fx
10	7	70
20	12	240
30	16	480
40	10	400
50	5	250
Total	$\Sigma f = N = 50$	$\Sigma fx = 1440$

$$AM(\bar{x}) = \frac{\Sigma fx}{N} = \frac{1440}{50} = 28.8$$

(B) Continuous frequency distribution (grouped data) :

In case of continuous frequency distribution, the variable is presented in the form of class intervals. Here the AM is obtained by taking the mid values of the class intervals and denoting them by x.

$$\therefore AM(\bar{x}) = \frac{\sum_{i=1}^n f_i x_i}{N}, \text{ where } x_i \text{ is the mid value of the } i^{\text{th}} \text{ class interval and is given by}$$

$$x_i = \frac{\text{Lower limit of } i^{\text{th}} \text{ class interval} + \text{Upper limit of } i^{\text{th}} \text{ class interval}}{2}$$

Example 3 : From the following data of the marks obtained by 30 students of a class, calculate the average marks.

Marks :	0–10	10–20	20–30	30–40	40–50
No. of Students:	3	6	10	7	4

Solution : Let ‘x’ denote the mid value of the class interval and ‘f’ the number of students. Now let us prepare the following table.

Marks	Mid value	No. of students	fx
C.I	x	f	
0–10	$\frac{0+10}{2} = 5$	3	15
10–20	$\frac{10+20}{2} = 15$	6	90

Marks	Mid value	No. of students	fx
20–30	$\frac{20+30}{2} = 25$	10	250
30–40	$\frac{30+40}{2} = 35$	7	245
40–50	$\frac{40+50}{2} = 45$	4	180
Total		$\sum_{\ell=1}^n f_i = N = 30$	$\sum_{\ell=1}^n f_i x_i = 780$

$$\text{Average} = \bar{x} \text{ (AM)} = \frac{\sum_{\ell=1}^n f_i x_i}{N} = \frac{780}{30} = 26$$

\therefore Average mark is 26.

4.5.2 PROPERTIES OF ARITHMETIC MEAN :

Property 1: The algebraic sum of the deviations from their arithmetic mean is zero. If mean of 'n' observations x_1, x_2, \dots, x_n is \bar{x} , then

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + \dots + (x_n - \bar{x}) = 0$$

$$\text{i.e., } \sum_{\ell=1}^n (x_i - \bar{x}) = 0$$

Proof: $(x_1 - \bar{x}) + (x_2 - \bar{x}) + \dots + (x_n - \bar{x})$
 $= (x_1 + x_2 + \dots + x_n) - (\bar{x} + \bar{x} + \dots \text{ n times})$

$$= \sum_{\ell=1}^n x^i - n\bar{x}$$

$$= n\bar{x} - n\bar{x} \left[\begin{array}{l} \because \bar{x} = \frac{\sum_{\ell=1}^n x}{n}, \quad \therefore n\bar{x} = \sum_{\ell=1}^n x \end{array} \right]$$

$$= 0.$$

Thus, the algebraic sum of deviations from mean is zero.

For a frequency distribution

$$\sum_{i=1}^n f_i(x_i - \bar{x}) = 0$$

Property 2 : If each observation from a given set of ‘n’ observations is increased or decreased by ‘a’ (constant) then the mean is also increased or decreased by ‘a’ ie., If the mean of ‘n’ observations x_1, x_2, \dots, x_n is \bar{x} , then the mean of the new observations–

$$(x_1 \pm a), (x_2 \pm a), \dots, (x_n \pm a) \text{ is } \bar{x} \pm a.$$

Property 3 : If each observation from a given set of ‘n’ observations is multiplied or divided by ‘a’ (a constant) then the mean is also multiplied or divided by ‘a’ ie., If the mean of ‘n’ observations x_1, x_2, \dots, x_n is \bar{x} , then the mean of new observations ax_1, ax_2, \dots, ax_n is $a\bar{x}$.

Again, the mean of new observations which we get by dividing x_1, x_2, \dots, x_n by ‘a’ ie, $\frac{x_1}{a}, \frac{x_2}{a}, \dots, \frac{x_n}{a}$ is $\frac{\bar{x}}{a}$.

Property 4 : If we are given the sizes and means of two series then the mean of the combined series can be obtained. If n_1 and n_2 are the sizes and \bar{x}_1 and \bar{x}_2 are the respective means of the two series then the mean \bar{x} of the combined series of sizes n_1+n_2 is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

Note :

The above result can be generalised to the case of more than two series. If we have ‘k’ series with respective sizes n_1, n_2, \dots, n_k and means $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$ then the combined mean is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \dots + n_k\bar{x}_k}{n_1 + n_2 + \dots + n_k}$$

$$= \frac{\sum_{i=1}^k n_i \bar{x}_i}{\sum_{i=1}^k n_i}$$

Property 5 : The sum of the squares of deviations of the given set of observations is minimum when taken from the AM. Symbolically, for a given frequency distribution, the sum, $S = \sum_{i=1}^n f_i(x_i - A)^2$, which represents the sum of the squares of deviations of given observation from any arbitrary value 'A' is minimum when $A = \bar{x}$.

Note :

If \bar{x} be AM of different values of the variate x, \bar{y} be the AM of the variate y and if their simple relation is $y = a + bx$, (where a and b are constant) then $\bar{y} = a + b\bar{x}$.

We have $y = a + bx$

Taking Σ both sides we get

$$\Rightarrow \Sigma y = \Sigma a + b \Sigma x$$

$$\Rightarrow \frac{\Sigma y}{n} = \frac{\Sigma a}{n} = \frac{b \Sigma x}{n} \quad [\text{dividing both sides by } n]$$

$$\Rightarrow \bar{y} = a + b\bar{x} \quad \left[\because \frac{\Sigma a}{n} = \frac{na}{n} = a \right]$$

Example 4: Average wage of male workers working in a factory is Rs.80, and the number of male workers is 100. The average wage of the female workers is Rs.70 and their number is 50. Find the average wage of all the workers taken together.

Solution : Here $n_1 = \text{No. of male workers} = 100$
 $n_2 = \text{No. of female workers} = 50$

$$\therefore \text{Total no. of workers} = n_1 + n_2 = 100 + 50 = 150$$

$$\bar{x}_1 = \text{Average wage of male workers} = \text{Rs.}80$$

$$\bar{x}_2 = \text{Average wage of female workers} = \text{Rs.}70$$

$$\bar{x} = \text{Average wage of all the workers} = ?$$

Now, we have

$$\begin{aligned} \bar{x} &= \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2} = \frac{100 \times 80 + 50 \times 70}{100 + 50} \\ &= \frac{8000 + 3500}{150} = 76.67 \end{aligned}$$

Average wage of all the workers is Rs.76.67.

4.5.3 MERITS AND DEMERITS OF ARITHMETIC MEAN

Merits of A.M. are :

- (i) It is rigidly defined.
- (ii) It is easy to understand and calculate.
- (iii) It is based on all the observations.
- (iv) It is suitable for further mathematical treatment.
- (v) It is least affected by sampling fluctuations.
- (vi) It is the best measure to compare two or more series.

Demerits of A.M. are :

- (i) It is very much affected by extreme observations (ie, very large or very small observations)
- (ii) It cannot be calculated if a single observation is missing or lost.
- (iii) It cannot be determined for qualitative data such as honesty, intelligence etc.
- (iv) It cannot be used in case of open end classes such as – less than 10, more than 30 etc.
- (v) It cannot be determined by inspection nor it can be located graphically.
- (vi) It may not be one of the values which the variable actually takes.

Uses of A.M. :

- (i) AM is used for calculating average like – average speed, average marks etc. Usually by average we mean arithmetic mean (A.M.)
- (ii) AM is widely used in business to find out the operation cost, profit per unit of article, output per man and per machine, average monthly income and expenditure etc.
- (iii) A.M. is considered to be the ideal average. It is also used in social and economic problems.

4.5.4 CALCULATION OF AM BY SHORT-CUT METHOD (Method of Assumed Mean) :

(i) Short-Cut Method for Ungrouped data :

The short-cut method is applied when the frequencies and the values of the variables are quite large. Here a provisional or assumed mean is taken which corresponds to the highest frequency or which corresponds to the middle value of a frequency distribution. Next we find the deviations of the variates from this assumed mean.

Then the A.M. is given by –

$$\bar{x} = A + \frac{\sum fd}{\sum f} = A + \frac{\sum fd}{N}$$

Where A = assumed mean

f = frequency

N = $\sum f$ = total frequency

d = x – A = deviations of any variate value from A.

Example 5:

Compute the mean from the following data of marks obtained by 50 students in Mathematics.

Marks :	10	20	30	40	50
No. of Students :	7	12	16	10	5

Solution : Let x denote the marks and 'f' denote the number of students.

Let us take the assumed mean A=30. For calculation of mean we prepare the following table

Marks x	No. of Students f	d = x – A = x – 30	fd
10	7	–20	–140
20	12	–10	–120
30	16	0	0
40	10	10	100
50	5	20	100
	$\sum f = N = 50$		$\sum fd = -60$

$$\begin{aligned}\bar{x} &= A + \frac{\sum fd}{N} \\ &= 30 + \frac{(-60)}{50} \\ &= 30 - \frac{60}{50} \\ &= 28.8\end{aligned}$$

Average marks is 28.8.

(ii) Short-cut method for grouped data (continuous series) :

For grouped data or continuous series we obtain the mid values of each class intervals which are denoted by x . Here an assumed mean is taken from the mid values which corresponds to the highest frequency or to the middle value of a frequency distribution. Next we find the deviations of mid values from the assumed mean. Then the A.M. is given by

$$\bar{x} = A + \frac{\sum fd}{N}$$

Where A = assumed mean

$d = x - A$ = deviations of mid-values from assumed mean

$N = \sum f$ = total frequency.

Example 6 :

From the following data of marks obtained by 50 students of a class, calculate the average marks.

Marks : 0–10 10–20 20–30 30–40 40–50

No. of Students : 7 12 16 10 5

Solution : For calculation of A.M. we prepare the following table.

Marks (C.I.)	Mid-value 'x'	No. of Students 'f'	d = x - A = x - 25	f.d.
0–10	5	7	-20	-140
10–20	15	12	-10	-120
20–30	25	16	0	0
30–40	35	10	10	100
40–50	45	5	20	100
		$\sum f = N = 50$		$\sum fd = -60$

Let the assumed mean $A=25$

$$\begin{aligned}\bar{x} &= A + \frac{\sum fd}{N} = 25 + \frac{(-60)}{50} \\ &= 25 - \frac{6}{5} = 25 - 1.2 = 23.8\end{aligned}$$

Average marks is 23.8.

4.5.5 CALCULATION OF A.M. BY STEP DEVIATION METHOD

(i) Step Deviation Method for Ungrouped Data :

In the step deviation method for ungrouped data, the only additional point is that in order to simplify calculations we take a common factor from the data. This common factor is equal to the common difference in the x values and is denoted by 'i'.

Here the deviation of variates 'x' from the assumed mean A are divided by the common factor and is denoted by 'd'.

The A.M. is denoted by the formula.

$$\bar{x} = A + \frac{\sum fd'}{N} \times i$$

where A = assumed mean

f = frequency

$$d' = \frac{x-A}{i}, i = \text{a common factor}$$

$$N = \sum f = \text{total frequency.}$$

Example 7:

Calculate by step-deviation method the A.M. of the following marks obtained by students in English.

Marks :	10	20	30	40	50	60
No. of Students :	4	6	10	12	8	5

Solution : Let the assumed mean = 40. For calculation of A.M. we prepare the following table.

Marks x	No. of Students f	d = x-A = x-40	d' = $\frac{d}{i} = \frac{d}{10}$	fd'
10	4	-30	-3	-12
20	6	-20	-2	-12
30	10	-10	-1	-10
40	12	0	0	0
50	8	10	1	8
60	5	20	2	10
Total	$\sum f = N = 45$			$\sum fd' = -16$

$$\begin{aligned}\bar{x} &= A + \frac{\sum fd'}{N} \times i \\ &= 40 + \frac{-16}{45} \times 10 \\ &= 40 - 3.55 \\ &= 36.45\end{aligned}$$

\therefore Average marks is 36.45.

(ii) Step-deviation method for grouped or continuous series :

When the class intervals in a grouped data are equal, then the calculation can be simplified further by taking a common factor. This common factor is the width of the class interval and is denoted by 'i'. The other steps in the calculation of A.M. remains the same as in ungrouped data.

Example 8 :

Find out the mean daily wages from the following distribution by step-deviation method.

Wages (Rs.) :	100–110	110–120	120–130	130–140	140–150	150–160
No. of Workers :	8	20	40	18	10	4

Solution : For calculation we prepare the following table.

Wages (C.I.)	Mid value (x)	No. of workers (f)	$d' = \frac{x-A}{i}$ $= \frac{x-125}{10}$	fd'
100–110	105	8	-2	-16
110–120	115	20	-1	-20
120–130	125	40	0	0
130–140	135	18	1	18
140–150	145	10	2	20
150–160	155	4	3	12
		$\Sigma f = N = 100$		$\Sigma fd' = 14$

Let the assumed mean be 125

$$\begin{aligned}
 \therefore \bar{x} &= A + \frac{\sum fd'}{N} \times i \\
 &= 125 + \frac{14}{100} \times 10 \\
 &= 125 + 1.4 \\
 &= 126.4
 \end{aligned}$$

The mean daily wages = Rs. 126.4.

Note :

If the data are given by the **inclusive method** like 10–19, 20–29, 30–39, it is not necessary to adjust the classes by finding the class boundaries for calculating A.M. To calculate A.M. we need the mid values of the classes which remain the same whether we make adjustments or not. It is also not necessary to re-arrange the series in an ascending order.

4.5.6 CALCULATION OF A.M. FROM CUMULATIVE FREQUENCY DISTRIBUTION

For calculation of A.M. from cumulative frequency distribution, we have to convert the cumulative frequency distribution to simple frequency distribution. After the conversion, we have to apply the formula to calculate the mean as it has been done in earlier example.

Example 9 :

The following are the marks obtained by the students of a class in statistics.

Find the average mark

Marks	No. of Students
less than 10	5
” ” 20	17
” ” 30	32
” ” 40	42
” ” 50	50

Solution :

First we convert the cumulative frequency (c.f.) to simple frequency and then we construct the grouped frequency distribution to find the A.M. as follows –

Here the width of the classes is 10, so the class intervals are taken as 0–10, 10–20,

Marks (C.I.)	Mid value (x)	No. of Students (f)	$d' = \frac{x-A}{i}$ $= \frac{x-25}{10}$	fd'
0-10	5	5	-2	-10
10-20	15	17-5=12	-1	-12
20-30	25	32-17=15	0	0
30-40	35	42-32=10	1	10
40-50	45	50-42=8	2	16
Total		$\Sigma f = N = 100$		$\Sigma fd' = 4$

Let the assumed mean be 25

$$\begin{aligned} \therefore \bar{x} &= A + \frac{\sum fd'}{N} \times i \\ &= 25 + \frac{4}{100} \times 10 \\ &= 25 + 0.4 \\ &= 25.4 \end{aligned}$$

The average mark is 25.4.

Example 10 :

Following in the distribution of marks out of 60 obtained by 50 students in a class. Calculate the mean.

Marks	No. of Students
Morethan 0	50
” ” 10	46
” ” 20	36
” ” 30	18
” ” 40	9
” ” 50	3

Solution :

First we convert the cumulative frequency (more than type) to simple frequency and then re-arrange the distribution as follows –

Marks (C.I.)	Mid value (x)	No. of Students (f)	$d' = \frac{x-A}{i}$ $= \frac{x-25}{10}$	fd'
0-10	5	50-46 = 4	-2	-8
10-20	15	46-36 = 10	-1	-10
20-30	25	36-18 = 18	0	0
30-40	35	18-9 = 9	1	9
40-50	45	9-3 = 6	2	12
50-60	55	3	3	9
		$\Sigma f = N = 50$		$\Sigma fd' = 12$

Let the assumed mean be 25

$$\begin{aligned} \therefore \bar{x} &= A + \frac{\Sigma fd'}{N} \times i \\ &= 25 + \frac{12}{50} \times 10 \\ &= 25 + 2.4 \\ &= 27.4 \end{aligned}$$

The average mark is 27.4.

Remark 1 : The AM from unequal width of class interval is obtained in the same way as it is obtained in case of equal width of class interval.

Remarks 2 : In case of open end classes, if we want to calculate mean we must close the ends of the class interval and then proceed in the usual manner.

Remarks 3 : If the class-intervals are of varying width then AM is not an appropriate measure of central tendency. In such cases, median and mode are used instead of mean.

Example on Calculation of missing frequency when A.M as given.

Example 11 : The AM of the following frequency distribution is 67.45 inches. Find the value of the missing frequency (f_3).

Height (in inches) :	60-62	63-65	66-68	69-71	72-74
No. of Persons :	5	54	f_3	81	24

Solution : Let us prepare the following table to calculate the missing frequency f_3 .

Height (C.I.)	Mid values (x)	No. of Persons (f)	fx
60–62	61	5	305
63–65	64	54	3456
66–68	67	f_3	$67f_3$
69–71	70	81	5670
72–74	73	24	1752
Total		$\Sigma f = 164 + f_3$	$11183 + 67f_3$

$$\bar{x} = \frac{\sum fx}{N}$$

$$67.45 = \frac{11183 + 67f_3}{164 + f_3}$$

$$\Rightarrow 67.45 (164 + f_3) = 11183 + 67f_3$$

$$\Rightarrow 11061.8 + 67.45f_3 = 11183 + 67f_3$$

$$\Rightarrow 67.45f_3 - 67f_3 = 11183 - 11061.8$$

$$\Rightarrow 0.45f_3 = 121.2$$

$$\therefore f_3 = \frac{121.2}{0.45} = 269.33$$

$$f_3 \simeq 269 \text{ Ans.}$$

Example on correction of incorrect value of a series when AM is given :

Formula for correct mean = $\frac{\text{Correct value of } \sum x}{n}$

Corrected $\sum x$ = Incorrect $\sum x$ – Incorrect observation + Correct observations.

Example 12 : The mean of 15 observations was found to be 20. Later on it was found that one observation 14 was misread as 40. Find the correct mean.

Solution : Incorrect mean (\bar{x}) = 20

Here n = 15

$$\bar{x} = \frac{\sum x}{n}$$

$$\Rightarrow n\bar{x} = \sum x$$

$$\begin{aligned} \therefore \text{Incorrect } \sum x &= n \times \text{Incorrect } \bar{x} \\ &= 15 \times 20 = 300 \end{aligned}$$

$$\begin{aligned} \text{Correct } \sum x &= \text{Incorrect } \sum x - \text{Incorrect observation} + \text{Correct observations} \\ &= 300 - 40 + 14 \end{aligned}$$

$$\text{Correct } \sum x = 274$$

$$\begin{aligned} \therefore \text{Correct } \bar{x} &= \frac{\text{Correct } \sum x}{n} \\ &= \frac{274}{15} \\ &= 18.266 \simeq 18.27 \text{ Ans.} \end{aligned}$$

4.5.7 WEIGHTED ARITHMETIC MEAN

While computing simple arithmetic mean, it is presumed that every item is of equal importance. Situations may arise where different items may vary in importance. In such cases, it becomes necessary to assign weights in proportion to the relative importance of the various items. We then compute the weighted arithmetic mean.

If w_1, w_2, \dots, w_n are the weights assigned to the values x_1, x_2, \dots, x_n respectively, then the weighted mean is defined as :

$$\text{Weighted AM} = \frac{w_1x_1 + w_2x_2 + \dots + w_nx_n}{w_1 + w_2 + \dots + w_n}$$

$$\bar{x}_w = \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$$

This is precisely same as the formula given in case of frequency distribution, with 'f' replaced by 'w'.

Example 13 : A candidate obtains the following percentages in an examination – English 60, Assamese 75, Mathematics 80, Physics 85 and Chemistry 62. Find the candidate's weighted AM if weights 1,2,1,3,3 respectively are allotted to the subjects.

Solution : Let 'x' denote the percentage of marks in the different subjects and 'w' denote the respective weight. Now we have the following table.

Subjects	% Marks (x)	Weights (w)	wx
English	60	1	60
Assamese	75	2	150
Mathematics	80	1	80
Physics	85	3	255
Chemistry	6	3	18.6
		$\Sigma f = 10$	$\Sigma wx = 731$

$$\begin{aligned} \text{Weighted mean} &= \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i} \\ &= \frac{731}{10} \\ &= 73.1 \text{ Ans.} \end{aligned}$$

EXERCISE 4 (A)

1. Define Average. Mention the objectives of average.
2. Discuss the characteristics of an ideal average.
3. What are the measures of central tendency? Why are they called measures of central tendency?
4. Mention the merits, demerits and uses of arithmetic mean.
5. Prove that the algebraic sum of the deviations of a given set of observations from their AM is zero.
6. If \bar{x} be AM of x variates, \bar{y} be AM of y variates and if $y=a+bx$, where a and b are constants, then show that $\bar{y} = a + b\bar{x}$.

7. Distinguish between simple and weighted average.
8. The weight of 6 persons are as follows (in Kg) 70, 42, 85, 75, 68, 55. Find the mean.
[Ans: 65.83 Kg.]
9. Calculate the AM of the following marks obtained by students in English.

Marks :	10	20	30	40	50	60
No. of Students :	6	4	10	12	8	4

[Ans : 35.6]

10. Determine the AM of the following distribution :

Height (in cm) :	130–134	135–139	140–144	145–149
Frequency :	5	15	28	24
Height (in cm) :	150–154	155–159	160–164	
Frequency :	17	10	1	

[Ans : 145.35 cm]

11. The AM of the values of a variable x is 20.
- (i) If each value is increased by 4, what will be the new AM?
- (ii) If each value is decreased by 5, what will be the new AM?
- (iii) If each value is multiplied by 2, what will be the new AM?
- (iv) If each value is divided by 4, what will be the new AM?

[Ans : (i) 24, (ii) 15, (iii) 40, (iv) 5]

12. Compute the AM of the following distribution –

Marks	No. of Students
less than 10	4
” ” 20	10
” ” 30	15
” ” 40	25
” ” 50	30
” ” 60	35
” ” 70	45
” ” 80	65

[Ans : 49.77]

13. The mean of the following frequency distribution is 50. But the frequencies f_1 and f_2 in the class 20–40 and 60–80 are missing. Find the missing frequencies.

Class :	0–20	20–40	40–60	60–80	80–100	Total
Frequency :	17	f_1	32	f_2	19	120

[Ans : $f_1 = 28, f_2 = 24$]

14. There are 50 students in a class of which 40 are boys and rest girls. The average weight of the class is 44Kg and the average weight of the girls is 40Kg. Find the average weight of the boys.

[Ans : 45Kg]

15. The mean of 200 observations was 50. Later on, it was found that two observations were misread as 92 and 8 instead of 192 and 88. Find the correct mean.

[Ans : 50.9]

16. A candidate obtains the following percentages in an examination – English 46%, Mathematics 67%, Statistics 72%, Economics 58% and Political Science 53%. It is agreed to double the weights to marks in English and Mathematics as compared to the other subjects. What is the weighted mean?

[Ans : 58.43]

4.6 GEOMETRIC MEAN (G.M.)

The geometric mean of a series of ‘n’ observations is defined as the n^{th} root of the product of the ‘n’ observations. Geometric mean is abbreviated as G.M.

Thus if x_1, x_2, \dots, x_n are ‘n’ observations then their geometric mean is given by :

$$\text{G.M.} = \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} = (x_1 \cdot x_2 \cdot \dots \cdot x_n)^{1/n} \dots \dots (1)$$

If there are two observations we take the square root of the product of the two observations as G.M.

eg. G.M. of 4 and 9 is $\sqrt{4 \times 9} = \sqrt{36} = 6$

Similarly, if there are three observations we take the cube root of the product of the three observations as the G.M. If the number of observations are greater than 3, then the computations becomes very tedious. In such cases G.M. can be calculated by taking the logarithm on both sides of (1)

$$\begin{aligned}\log(\text{GM}) &= \frac{1}{n} \log(x_1 \cdot x_2 \cdot \dots \cdot x_n) \\ &= \frac{1}{n} [\log x_1 + \log x_2 + \dots + \log x_n] \\ &= \frac{1}{n} \sum_{\ell=1}^n \log x_i\end{aligned}$$

$$\therefore \text{GM} = \text{Antilog} \left[\frac{1}{n} \sum_{\ell=1}^n \log x_i \right]$$

4.6.1 CALCULATION OF G.M. FOR INDIVIDUAL SERIES

Following are the steps to calculate the G.M. for individual series.

- (i) Find $\log x$, where x is the value of the observations.
- (ii) Add all the values of $\log x$ ie, $\sum \log x$.
- (iii) $\sum \log x$ is divided by the total number of observations, which gives $\frac{\sum \log x}{n}$.
- (iv) The antilog of $\frac{\sum \log x}{n}$ gives the required G.M.

Example 14 : Calculate the GM of the following

50, 72, 54, 82, 93.

Solution : For computation of G.M. let us prepare the following table :

x	log x
50	1.6990
72	1.8573
54	1.7324
82	1.9138
93	1.9685
	$\sum \log x = 9.1710$

$$\begin{aligned}\text{G.M.} &= \sqrt[5]{50 \times 72 \times 54 \times 82 \times 93} \\ &= (50 \times 72 \times 54 \times 82 \times 93)^{1/5}\end{aligned}$$

$$\begin{aligned}
 \text{or log GM} &= \frac{1}{5}[\log 50 + \log 72 + \log 54 + \log 82 + \log 93] \\
 &= \frac{1}{5}[1.6990 + 1.8573 + 1.7324 + 1.9138 + 1.9685] \\
 &= \frac{1}{5} \times 9.1710 \\
 &= 1.8342. \\
 \therefore \text{G.M.} &= \text{Antilog } 1.8342 \\
 &= 68.29 \text{ Ans.}
 \end{aligned}$$

4.6.2 CALCULATION OF G.M.

(i) Calculation of g.m. for ungrouped frequency distribution :

If the observations x_1, x_2, \dots, x_n occur with frequencies f_1, f_2, \dots, f_n respectively, and the total frequency, $\sum f = N$, then the G.M. in case of ungrouped frequency distribution is given by

$$\begin{aligned}
 \text{G.M.} &= [(x_1 \cdot x_1 \dots \dots \text{f}_1 \text{ times}) \times (x_2 \cdot x_2 \dots \dots \text{f}_2 \text{ times}) \times \dots \times (x_n \cdot x_n \dots \dots \text{f}_n \text{ times})]^{1/N} \\
 \therefore \text{G. M.} &= [x_1^{f_1} \cdot x_2^{f_2} \dots \dots x_n^{f_n}]^{1/N}
 \end{aligned}$$

Taking log on both sides we get

$$\begin{aligned}
 \log \text{GM} &= \frac{1}{N} \log [x_1^{f_1} \cdot x_2^{f_2} \dots \dots x_n^{f_n}] \\
 &= \frac{1}{N} [f_1 \log x_1 + f_2 \log x_2 + \dots \dots + f_n \log x_n] \\
 &= \frac{1}{N} \sum_{i=1}^n f_i \log x_i \\
 \therefore \text{G. M.} &= \text{Anti log} \left[\frac{1}{N} \sum_{i=1}^n f_i \log x_i \right]
 \end{aligned}$$

Example 15 : Find the G.M. from the following data.

Marks :	5	15	25	35	45
No. of Students :	5	7	15	25	8

For calculation of G.M. we prepare the following table.

x	f	log x	f log x
5	5	0.6990	3.4950
15	7	1.1761	8.2327
25	15	1.3979	20.9685
35	25	1.5441	38.6025
45	8	1.6532	13.2256
	$\Sigma f = N$ = 60		$\Sigma f \log x$ = 84.5243

$$\begin{aligned}
 \text{G.M.} &= \text{Anti log} \left[\frac{1}{N} \Sigma f \log x \right] \\
 &= \text{Anti log} \left[\frac{84.5243}{60} \right] \\
 &= \text{Anti log } 1.4087 = 25.63 \\
 &= 25.63 \text{ marks Ans.}
 \end{aligned}$$

(ii) Calculation of g.m. for grouped or continuous frequency distributions :

For grouped or continuous frequency distribution, the values of x are the mid values of the corresponding classes.

The steps to calculate GM for grouped frequency distribution are same as that of ungrouped frequency distributions.

4.6.3 COMBINED GEOMETRIC MEAN

If G_1 and G_2 are the geometric means of two series of sizes n_1 and n_2 respectively, then the GM of the combined series of size n_1+n_2 is the Antilog (log GM). Where

$$\log \text{G.M.} = \frac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2}$$

The above result can be generalised to the case of 'k' series as follows – If G_1, G_2, \dots, G_k are the GM of the 'k' series of sizes n_1, n_2, \dots, n_k respectively, then the G.M. of the combined series of size $n_1+n_2+\dots+n_k$ is given by Antilog (log G.M.), where

$$\log \text{G.M.} = \frac{n_1 \log G_1 + n_2 \log G_2 + \dots + n_k \log G_k}{n_1 + n_2 + \dots + n_k}$$

$$= \frac{\sum_{k=1}^n n_k \log G_k}{\sum_{k=1}^n n_k}$$

$$\therefore \text{G.M.} = \text{Anti log} \frac{\sum_{k=1}^n n_k \log G_k}{\sum_{k=1}^n n_k}$$

4.6.4 WEIGHTED GEOMETRIC MEAN :

If the different observations $x_i (i=1,2,\dots,n)$ are not of equal importance and are assigned different weights say $w_i (i=1,2,\dots,n)$ respectively, according to their degree of importance then their weighted G.M. is defined as :

$$\text{G.M (w)} = \left[x_1^{w_1} \cdot x_2^{w_2} \cdot \dots \cdot x_n^{w_n} \right]^{1/N}$$

where $N = w_1 + w_2 + \dots + w_n = \sum_{\ell=1}^n w_i$

$$\therefore \log \text{G.M (w)} = \frac{1}{N} [w_1 \log x_1 + w_2 \log x_2 + \dots + w_n \log x_n]$$

$$= \frac{1}{\sum_{\ell=1}^n w_i} \sum_{\ell=1}^n w_i \log x_i$$

$$\therefore \text{G.M (w)} = \text{Antilog} \left[\frac{\sum_{\ell=1}^n w_i \log x_i}{\sum_{\ell=1}^n w_i} \right]$$

4.6.5 MERITS, DEMERITS AND USES OF G.M.**The merits of G.M. are :**

- (i) It is rigidly defined.
- (ii) It is based on all the observations.
- (iii) It is suitable for further mathematical treatment.
- (iv) It is less affected by extreme observations.
- (v) It is not affected much by fluctuations of sampling.
- (vi) It is suitable when large weights have to be given to small items and small weight to the large items.

The Demerits of G.M. are :

- (i) It is difficult to understand and calculate for a non-mathematical person.
- (ii) The value of G.M. cannot be calculated when any of the observation is zero or negative.

Uses of G.M. are :

- (i) G.M. is used in the construction of Index Number.
- (ii) G.M. is highly useful in averaging ratios, percentages and rates of increase between two periods.
- (iii) G.M. is used in economic and social sciences where we want to give more importance to smaller items.

EXERCISE 4 (B)

1. Define simple and weighted geometric mean.
2. State the merits, demerits and uses of G.M.
3. Find the G.M. of – (i) 2 and 8; (ii) 3, 9 and 27.
4. Compute the G.M. of the following data–
10, 110, 120, 50, 52, 80, 37, 60

[Ans: 52.84]

5. Find the G.M. from the following data :

Marks :	10	15	25	40	50
No. of Students :	4	6	10	7	3

[Ans: 23.89]

6. Find the G.M. from the following data :

C.I. :	0–10	10–20	20–30	30–40	40–50
f:	4	8	10	6	7

[Ans: 22.06]

7. Three sets of data contain 8, 7 and 5 observations and their geometric means are 8.52, 10.12 and 7.75 respectively. Find the combined G.M. of 20 observations.

[Ans: 8.835]

8. Find the G.M. of 111, 171, 191, 212 having weights 3, 2, 4 and 5 respectively.

[Ans: 173.4]

9. The rate of increase in population of a country during the last three decades are 5%, 8% and 12%. Find the average rate of growth during the last three decades.

[Ans: 8.2%]

10. A given machine is assumed to depreciate 40% in value in the first year, 25% in the 2nd year and 10% per year for the next three years, each percent being calculated on the diminishing value. What is the average depreciation recorded on the diminishing value for the period of 5 years.

[Ans: 15.85%]

4.7 HARMONIC MEAN (H.M.)

The harmonic mean abbreviated as H.M., is defined as the reciprocal of the arithmetic mean of the reciprocal of the individual observations.

Thus if x_1, x_2, \dots, x_n are 'n' observations then their H.M. is given by

$$\begin{aligned} \text{H.M.} &= \frac{1}{\frac{1}{n} \left[\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right]} = \frac{1}{\frac{1}{n} \sum_{i=1}^n \frac{1}{x_i}} \\ &= \frac{n}{\sum_{i=1}^n \left(\frac{1}{x_i} \right)} \end{aligned}$$

4.7.1 CALCULATION OF H.M. FOR INDIVIDUAL SERIES :

Following are the steps to calculate H.M. for an individual series

- (i). Find the reciprocal of each observation ie, $\frac{1}{x}$.

(ii) Find the reciprocals of all the observations ie, $\sum_{\ell=1}^n \frac{1}{x_i}$.

(iii) Then apply the formula H.M. = $\frac{n}{\sum_{\ell=1}^n \left(\frac{1}{x_i}\right)}$

where 'n' is the number of observations.

Example 16: Find the H.M. of 2, 4, 6, 8.

Solution : H.M. = $\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}$

$$= \frac{4}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}}$$

$$= \frac{4}{\frac{12+6+4+3}{24}}$$

$$= \frac{4}{\frac{25}{24}}$$

$$= 3.84$$

Ans: 3.84

4.7.2 CALCULATION OF H.M.

(i) For ungrouped frequency distribution :

If the observations x_1, x_2, \dots, x_n occur with frequencies f_1, f_2, \dots, f_n respectively, and the total frequency ie, $\sum_{\ell=1}^n f_i = N$. then H.M. is given by

$$\text{H.M.} = \frac{f_1 + f_2 + \dots + f_n}{\left[\frac{f_1}{x_1} + \frac{f_2}{x_2} + \dots + \frac{f_n}{x_n} \right]}$$

$$= \frac{N}{\sum_{\ell=1}^n \left(\frac{f_i}{x_i} \right)}$$

Example 17 : Obtain the H.M. of the following data :

x :	10	20	30	40	50
f :	5	6	9	4	1

Solution : For calculation of H.M. let us prepare the following table.

x	f	$\frac{1}{x}$	$f \times \frac{1}{x}$
10	5	0.100	0.500
20	6	0.050	0.300
30	9	0.033	0.297
40	4	0.025	0.100
50	1	0.020	0.020
Total	$\Sigma f = N$ = 25		$\Sigma \frac{f_i}{x_i} = 1.217$

$$\begin{aligned} \therefore \text{H.M.} &= \frac{N}{\sum_{i=1}^n \left(\frac{f_i}{x_i} \right)} \\ &= \frac{25}{1.217} \\ &= 20.54 \end{aligned}$$

Ans: 20.54.

(ii) Calculation of H.M. for grouped or Continuous frequency distributions :

For calculating H.M. in a continuous frequency distribution, the procedure is the same as applied to ungrouped frequency distribution. The only difference is that here we take the reciprocal of the mid-points of the close intervals as variate values. Thus in continuous or grouped frequency distribution H.M. is computed by applying the following formula

$$\text{H.M.} = \frac{N}{\sum_{i=1}^n \left(\frac{f_i}{x_i} \right)}$$

where x_i is the mid-point of the i^{th} class interval and $N = \sum_{i=1}^n f_i$.

4.7.3 WEIGHTED HARMONIC MEAN

Weighted H.M. is calculated with the help of the following formula.

$$\begin{aligned} \text{H.M.}(w) &= \frac{\sum_{i=1}^n w_i}{\left(\frac{1}{x_1} \times w_1\right) + \left(\frac{1}{x_2} \times w_2\right) + \dots + \left(\frac{1}{x_n} \times w_n\right)} \\ &= \frac{\sum_{i=1}^n w_i}{\sum_{i=1}^n \left(\frac{w_i}{x_i}\right)} \end{aligned}$$

4.7.4 MERITS, DEMERITS AND USES OF H.M. MERITS OF H.M. ARE :

- (i) It is rigidly defined.
- (ii) It is based on all the observations.
- (iii) It is suitable for further mathematical treatment.
- (iv) It is not affected very much by fluctuations of sampling.
- (v) It is useful in averaging special types of rate and ratios where time factor is variable and the act being performed remains constant.

Demerits of H.M. :

- (i) It is not easy to understand and calculate.
- (ii) Its value cannot be computed when there are negative observations or when one or more observations are zero.
- (iii) It is not very useful for analysis of economic data as it given greater weightage to smaller items.

Uses of H.M.

Harmonic mean is useful in problems in which values of a variable are compared with a constant quantity of another variable ie, rates, time, distance covered within certain time and quantities purchased or sold per unit etc.

Problem 18: A man travels from one place to another, a distance of 120km at an average speed of 40km/hr. He then makes the return journey at average speed of 30km/hr. What is his average speed over the entire distance?

Solution : H.M. would be suitable to calculate the average speed.

$$\therefore \text{H.M.} = \frac{2}{\frac{1}{40} + \frac{1}{30}} = \frac{2}{\frac{7}{120}} = 34.29.$$

Ans : 34.29 km/hr.

Remark : The above e.g., clearly shows that when distances are the same for the two speeds, H.M. given the correct answer, but when time of the journey are given the A.M. of the rates of speed gives the correct answer.

EXERCISE 4 (C)

1. Define H.M. what are its uses?
2. Mention the merits and demerits of H.M.
3. Find the H.M. of the following
 - (i) 5, 8, 10, 15, 24 [Ans: 9.38]
 - (ii) $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$ [Ans: $\frac{2}{n+1}$]
 - (iii) $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{10}$ [Ans: 0.18]
4. A person travelled 10 hrs at 50km/hr and again 14 hrs at 40km/hr, find the average speed. [Ans: 44.17km/hr]
5. A car covers four sides of a square at speed of 10, 20, 30 and 40km/hr respectively. What is the average speed of the car around the square. [Ans: 19.20km/hr]
6. An investor buys Rs.20,000 worth of shares of a company each month. During the first three month he bought the shares at prices Rs.120, Rs.160 and Rs.210. After three months what is the average price paid by him for the shares. [Ans: Rs.166.66]
7. A cyclist covers his first 5 km at an average speed of 10km/hr, another 3 km at 8km/hr and the last 2 km at 5km/hr. Find the average speed for the entire journey. [Ans: 7.84 km/hr]

8. Obtain the H.M. from the following data.

Marks :	10	20	30	40	50
No. of Students :	4	5	8	2	1

[Ans: 20.33]

9. Compute the H.M. from the following data.

Class Interval (C.I.) :	0–8	8–16	16–24	24–32	32–40	40–48
Frequency (f) :	8	7	6	24	15	7

[Ans: 25.40]

10. A man makes a trip which entails travelling 1500 km by train at an average speed of 60km/hr, 300 km by boat at an average speed of 25 km/hr, and 90 km by car at 30 km/hr. What is the average speed for the entire journey.

[Ans: 47.25 km/hr.]

4.8 RELATIONSHIP AMONG AM, GM AND H.M.

For any set of observations, AM, GM and HM are related to each other in the relationship

$$AM \geq GM \geq HM$$

The sign of equality holds of all the observations are identical.

For the observations x_1 & x_2 , we have

$$(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$$

$$\Rightarrow (\sqrt{x_1})^2 - 2\sqrt{x_1}\sqrt{x_2} + (\sqrt{x_2})^2 \geq 0$$

$$\Rightarrow x_1 + x_2 \geq 2\sqrt{x_1x_2}$$

$$\Rightarrow \frac{x_1 + x_2}{2} \geq \sqrt{x_1x_2}$$

$$\Rightarrow AM \geq GM \text{ (This is true for any number of observations)}$$

Again for $\frac{1}{x_1}$ and $\frac{1}{x_2}$ observations, we have

$$\left(\sqrt{\frac{1}{x_1}} - \sqrt{\frac{1}{x_2}} \right)^2 \geq 0$$

$$\Rightarrow \frac{1}{x_1} + \frac{1}{x_2} - 2\sqrt{\frac{1}{x_1} \frac{1}{x_2}} \geq 0$$

$$\Rightarrow \frac{1}{x_1} + \frac{1}{x_2} \geq 2\sqrt{\frac{1}{x_1 x_2}}$$

$$\Rightarrow \frac{1}{x_1} + \frac{1}{x_2} \geq \frac{2}{\sqrt{x_1 x_2}}$$

$$\Rightarrow \sqrt{x_1 x_2} \geq \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}$$

$\Rightarrow GM \geq HM$ (True for any number of observations)

$\therefore AM \geq GM \geq HM$

We also have $AM \times HM = (GM)^2$

This relationship holds only for a pair of observations.

Let the pair of observations be x_1 and x_2

$$\begin{aligned} \therefore AM \times HM &= \frac{x_1 + x_2}{2} \times \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} \\ &= \frac{x_1 + x_2}{2} \times \frac{2x_1 x_2}{x_1 + x_2} \\ &= x_1 x_2 = (\sqrt{x_1 x_2})^2 = (GM)^2 \\ \therefore AM \times HM &= (GM)^2 \end{aligned}$$

Problem 19 : Find the AM, GM and HM for the numbers 2, 4, 6 and show that $AM > GM > HM$.

Solutions : The three observations are 2, 4 & 6.

$$\therefore AM = \frac{2+4+6}{3} = \frac{12}{3} = 4$$

$$GM = \sqrt[3]{2 \times 4 \times 6} = (2 \times 4 \times 6)^{\frac{1}{3}}$$

$$\therefore \log GM = \frac{1}{3}[\log 2 + \log 4 + \log 6]$$

$$= \frac{1}{3}[0.3010 + 0.6021 + 0.7782]$$

$$\begin{aligned}
 &= \frac{1}{3}[1.6813] \\
 &= 0.5604 \\
 \therefore \text{GM} &= \text{Anti log}(0.5604) \\
 &= 3.634 \\
 \text{HM} &= \frac{3}{\frac{1}{2} + \frac{1}{4} + \frac{1}{6}} \\
 &= \frac{3}{\frac{6+3+2}{12}} = \frac{3}{\frac{11}{12}} = \frac{36}{11} = 3.27
 \end{aligned}$$

Now, we have AM = 4, GM = 3.634 & HM = 3.27

$\therefore \text{AM} > \text{GM} > \text{HM}$.

Problem 20 : If the AM and GM of two numbers are 10 and 8 respectively, find their HM and also the numbers.

Solution : We are given AM = 10 & GM = 8.

We know that $\text{AM} \times \text{HM} = (\text{GM})^2$

$$\text{or, } 10 \times \text{HM} = 8^2$$

$$\begin{aligned}
 \text{or, } \text{H.M.} &= \frac{64}{10} \\
 &= 6.4
 \end{aligned}$$

Let the two numbers be x and y.

$$\text{Now, } \text{AM} = \frac{x+y}{2} = 10$$

$$\Rightarrow \text{AM} = 20$$

$$\text{GM} = \sqrt{xy} = 8$$

$$\Rightarrow xy = 64$$

We know that $(x-y)^2 = (x+y)^2 - 4xy$

$$\Rightarrow (x-y)^2 = (20)^2 - 4 \times 64$$

$$\Rightarrow (x-y)^2 = 144$$

$$\therefore (x-y) = 12.$$

Solving $x+y = 20$ and $x-y = 12$ we get

$$2x = 32$$

$$\therefore x = 16$$

$$\therefore y = 20 - 16 = 4$$

Hence the numbers are 16 & 4.

EXERCISE 4 (D)

1. Find the AM, GM & HM for the numbers 4, 6, 8 and show that $AM > GM > HM$.
[Ans: Am=6, GM=5.755 & HM=5.538]
2. The AM and GM of two observations are 9 and 7.2 respectively. Find the two observations and also the H.M.
[Ans: Nos are 14.4 & 3.6, HM=5.76]
3. GM and HM of two numbers are 6 and 5.5 respectively, find AM.
[Ans: 6.5]
4. AM and HM of two numbers are 10 and 6.4 respectively. Find the numbers and their GM.
[Ans: Nos. are 16 & 4, GM=8]

4.9 MEDIAN

If a set of data is arranged in ascending or descending order of magnitude then the middle value which divides the set into two equal parts is the median. Median is known as positional average. According to Prof. L.R. Corner- "The median is that value of the variable which divides the group into two equal parts, one part comprising all values greater and the other all values less than median.

For eg, if the marks of seven students are given as below : 60, 62, 65, 70, 73, 75, 80.
Then median is given by 70.

4.9.1 CALCULATION OF MEDIAN :**(i) For Individual series :**

The 'n' values of the variable are arranged in ascending or decending order of magnitude, then

Case I– when 'n' is odd.

Median is the value of $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation

Case II– when 'n' is even

There will be two middle terms $\frac{n}{2}$ and $\frac{n}{2} + 1$, then median is the average of these two terms i.e.,

$$\text{Median} = \frac{\left(\frac{n}{2}\right)^{\text{th}} \text{ term} + \left(\frac{n}{2} + 1\right)^{\text{th}} \text{ term}}{2}$$

Example 21 : The marks scored by 5 students are 40, 42, 38, 45, 50. Find the median.

Solution : Let us arrange the values in ascending order of magnitude – 38, 40, 42, 45, 50
Here n=5 ie, odd number

$$\begin{aligned} \therefore \text{Median} &= \left(\frac{n+1}{2}\right)^{\text{th}} \text{ value} \\ &= \left(\frac{5+1}{2}\right)^{\text{th}} \text{ value} \\ &= 3^{\text{rd}} \text{ value} \\ &= 42 \\ \therefore \text{Median} &= 42 \text{ marks.} \end{aligned}$$

Example 22: Find the median of the following marks obtained by 6 students 35, 38, 40, 42, 45, 50.

Solution : Here n=6, ie, even

$$\begin{aligned} \therefore \text{Median} &= \text{average of } \left(\frac{n}{2}\right)^{\text{th}} \text{ and } \left(\frac{n}{2} + 1\right)^{\text{th}} \text{ value} \\ &= \text{average of } \left(\frac{6}{2}\right)^{\text{th}} \text{ and } \left(\frac{6}{2} + 1\right)^{\text{th}} \text{ value} \end{aligned}$$

= average of 3rd and 4th value

$$= \frac{40 + 42}{2}$$

$$= 41$$

∴ Median = 41 marks.

Example 23 : Find the median for the following frequency distribution.

Marks :	10	20	30	40	50
No. of Students :	4	6	16	7	2

Solution : Let us prepare the following table for calculation of median.

Marks	No. of Students (f)	c.f.
10	4	4
20	6	10
30	16	26
40	7	33
50	2	35
Total	$\Sigma f = N = 35$	

$$\text{Now, } \frac{N+1}{2} = \frac{35+1}{2} = \frac{36}{2} = 18$$

The cf just greater than 18 is 26. The value corresponding to c.f 26 is 30.

∴ Hence the median mark is 30.

(ii) For Simple Frequency Distribution (Ungrouped Data)

Here to calculate median we follow the following steps :

1. Arrange the values of the variable in ascending or descending order of magnitude.
2. Calculate the cumulative frequency less than type.
3. Find out the cumulative frequency (cf). Which is next higher or equal to $\frac{N+1}{2}$, where $N = \Sigma f$ (total frequency).
4. The value of the variable which corresponds to the c.f. is the required median.

(iii) For Continuous series (grouped frequency distribution)

In this case, for computation of median, we follow the following steps.

1. Arrange the class interval in ascending order.
2. Find the c.f. for different classes.
3. Find the value of $\frac{N}{2}$
[as in continuous series $\frac{N}{2}$ divides the area of the curve into two equal parts]
4. Find the c.f. which is equal to or next higher to $\frac{N}{2}$.
5. The class interval corresponding to this c.f. is called the median class.
6. Then use the following formula to find median

$$\text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Where L = lower limit of the median class.

$N = \sum f$ = total frequency.

c.f. = cumulative frequency of the class preceding the median class.

f = frequency of the median class.

i = width of the median class.

Example 24 : Calculate the median from the following data.

Class Interval :	10–20	20–30	30–40	40–50	50–60	60–70
Frequency :	10	18	30	27	15	8

Solution : Let us prepare the following table to compute median.

Class Interval	Frequency (f)	c.f.
10–20	10	10
20–30	18	28
30–40	30	58
40–50	27	85
50–60	15	100
60–70	8	108
Total	$\sum f = N = 108$	

$$\begin{aligned}\text{Median} &= \text{value of } \left(\frac{N}{2}\right)^{\text{th}} \text{ item} \\ &= \text{value of } \left(\frac{108}{2}\right)^{\text{th}} = 54^{\text{th}} \text{ item}\end{aligned}$$

The c.f. just greater than 54 is 58.

∴ Median lies in the class 30–40

$$\text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Here $L = 30$, $\frac{N}{2} = 54$, $\text{c.f.} = 28$, $f = 30$, $i = 10$

$$\begin{aligned}\therefore \text{Median} &= 30 + \frac{54 - 28}{30} \times 10 \\ &= 30 + 8.67 = 38.67\end{aligned}$$

Ans: 38.67.

Remarks :

1. If the class-intervals are in inclusive form, they must be converted into exclusive form before finding the median.
2. If the observation are increased or decreased by a fixed amount, then median is also increased or decreased by the same amount.
3. When the class interval are unequal it is not required to make them equal to compute median.
4. In case of open-end classes, the class-interval is not required to be closed as to compute median we require only the median class.

4.9.2 MERITS, DEMERITS AND USES OF MEDIAN

Merits of Median are as follows:

- (i) It is rigidly defined.
- (ii) It is easy to understand and calculate.
- (iii) It is not affected by extreme observations.
- (iv) It can be computed for a distribution with open-end classes.
- (v) It can be located by simple inspection and can be determined graphically.

Demerits of Medians :

- (i) It is not based on all the observations.
- (ii) It is not suitable for further mathematical treatments.
- (iii) It is affected more than mean by fluctuations of sampling.
- (iv) To compute median, it is necessary to arrange the data in ascending or descending order of magnitude.

Uses of Medians :

- (i) It is the only average to be used while dealing with qualitative characteristics like honesty, beauty, etc.
- (ii) It is used when extreme observations are present in the data.
- (iii) It is used to determine the average of a distribution with open end classes.
- (iv) In case of income distribution the use of median gives the best result.

Example 25: From the data given below calculate the median.

Marks : (less than)	10	20	30	40	50
No. of Students :	5	9	15	18	20

Solution : First we change the given cumulative distribution to single frequency distribution as follows :

C.I.	f	c.f.
0–10	5	5
10–20	$9-5 = 4$	9
20–30	$15-9 = 6$	15
30–40	$18-15 = 3$	18
40–50	$20-18 = 2$	20
Total	$\Sigma f = N = 20$	

$$\text{Median} = \text{value of } \left(\frac{N}{2}\right)^{\text{th}} \text{ item}$$

$$= \text{value of } \left(\frac{20}{2}\right) = 10^{\text{th}} \text{ item}$$

The c.f. next higher to 10 is 15

\therefore Median lies in class 20–30

$$\therefore \text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Here $L = 20$, $\frac{N}{2} = 10$, $\text{c.f.} = 9$, $f = 6$, and $i = 10$

$$\begin{aligned} \therefore \text{Median} &= 20 + \frac{10-9}{6} \times 10 \\ &= 20 + 1.67 = 21.67 \end{aligned}$$

Median Mark = 21.67.

Example 26 : Median of the following distribution in 24.5 find f_2 .

Class interval :	10–14	15–19	20–24	25–29	30–34	35–39
Frequency :	5	f_2	4	8	4	3

Solution : The data is in inclusive form, so first we convert it into exclusive form. Now let us prepare the following table.

C.I.	f	c.f.
9.5–14.5	5	5
14.5–19.5	f_2	$5+f_2$
19.5–24.5	4	$9+f_2$
24.5–29.5	8	$17+f_2$
29.5–34.5	4	$21+f_2$
34.5–39.5	3	$24+f_2$
	$\Sigma f = N = 24 + f_2$	

Since the median is 24.5, it lies in the class 24.5–29.5

$$\therefore \text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Here $L = 24.5$, $\frac{N}{2} = \frac{24+f_2}{2}$, $\text{c.f.} = 9+f_2$, $i = 5$, $f = 8$ and median = 24.5

$$\therefore 24.5 = 24.5 + \frac{\frac{24+f_2}{2} - 9+f_2}{8} \times 5$$

$$\Rightarrow 0 = \frac{24+f_2 - 18 + 2f_2}{2} \times 5$$

$$\Rightarrow 0 = 5(6-f_2)$$

$$\Rightarrow 5f_2 = 30$$

$$\therefore f_2 = 6. \text{ Ans.}$$

4.9.3 GRAPHICAL METHOD OF DETERMINING MEDIAN :

There are two methods of determining median graphically.

Method I : Let us draw two cumulative frequency curves or ogives, one by ‘less than’ method and the other by ‘more than’ method.

From the point where both these curves intersect each other we draw a perpendicular on the x-axis. The point where this perpendicular touches the x-axis gives the value of the median.

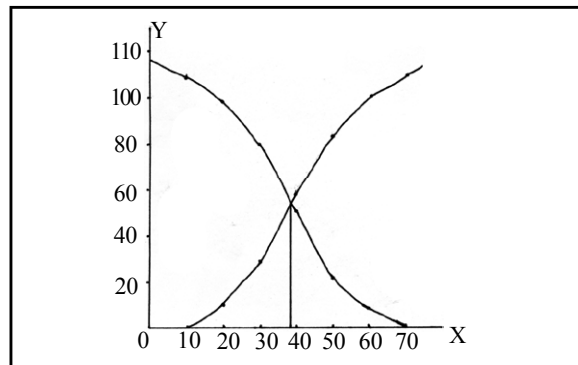
Method II : Let us draw one ogive or cumulative frequency curve by less than method. Now let us determine the median value by the formula, Median = value of $\left(\frac{N}{2}\right)^{\text{th}}$ observations. Let us locate this value $\frac{N}{2}$ on the y-axis and from it let us draw a perpendicular on the ogive. From the point where it meets the ogive we draw another perpendicular on the x-axis, and the point where it meets the x-axis is the median.

This will be clear from the example 24 done earlier.

Let us prepare the following cumulative frequency distribution table :

C.I.	F	c.f.(less than)	c.f.(more than)
10–20	10	10	108
20–30	18	28	98
30–40	30	58	80
40–50	27	85	50
50–60	15	100	23
60–70	8	108	8
Total	$\Sigma f = N = 108$		

Method I

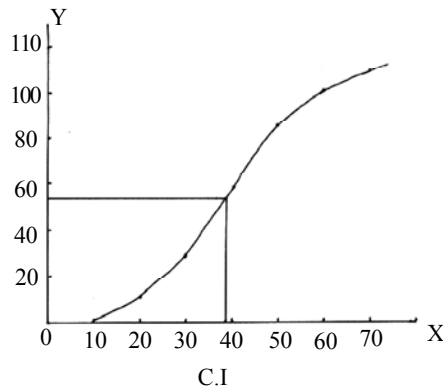


C.I

It is clear from the above graph that median value is 39 (approx), which can be verified with the solved example.

Method II : Median = value of $\left(\frac{108}{2}\right)^{\text{th}}$ item
 = value of 54th item

New taking 54 on the y-axis we get.



Here also it is clear from the above graph that the median value is 39 (approx).

4.10 PARTITION VALUES

Partition values divides a series into number of equal parts. Median is a partition value which divides a distribution into two equal parts. Quartiles, decides and percentites are also partition values which can be determined in the same way as median. The only difference is in their location.

Quartiles divide a series into four equal parts. There are three quartiles denoted by Q_1 , Q_2 and Q_3 .

Q_1 known as the first quartile or lower quartile divides a series in such a way that 25% of observations have a value less than Q_1 and 75% have a value more than Q_1 .

Q_2 known as the second quartile has the same number of observations above and below it. So it is same as the median value.

Q_3 known as the third quartile or Upper quartile divides the series in such a way that 75% of observations have a value less than Q_3 and 25% have a value more than Q_3 .

Deciles, divide a series into ten equal parts. There are nine deciles denoted by D_1 , D_2 , D_3 ,....., D_9 . Between any two consecutive decides there are 10% of the total observations. For eg, D_1 or first decile has 90% of the observations above it and 10% of observations below it. D_5 or fifth decile coincides with the median as it has 50% of the observations below and above it.

Percentiles divide a series into hundred equal parts. There are ninety-nine percentiles denoted by $P_1, P_2, P_3, \dots, P_{99}$. Between any two consecutive percentile there are 1% of the total observations. For eg, P_{30} or the 30th percentile has 30% observations below P_{30} and 70% of the observations above it. P_{50} coincides with the median as it has 50% of observations below and above it.

4.10.1 CALCULATION OF QUARTILES, DECILES AND PERCENTILES :

A. For simple data or series of individual observations

First the data are to be arranged in ascending order of magnitude, then

$$Q_k = \text{value of } \frac{k(n+1)}{4}^{\text{th}} \text{ observation, where } k = 1, 2, 3.$$

$$D_k = \text{value of } \frac{k(n+1)}{10}^{\text{th}} \text{ observation, where } k = 1, 2, \dots, 9.$$

$$P_k = \text{value of } \frac{k(n+1)}{100}^{\text{th}} \text{ observations, where } k = 1, 2, \dots, 99.$$

Example 27 : Find Q_1, Q_2, Q_3, D_6 and P_{75} from the following data of marks obtained out of 50 in English by 10 students.

Solution : First let us arrange the data in ascending order of magnitude

Marks : 11, 12, 14, 18, 22, 26, 30, 32, 35, 41

Here we have $n = 10$.

$$Q_1 = \text{value of } \left(\frac{n+1}{4}\right)^{\text{th}} \text{ observation.}$$

$$= \text{value of } \left(\frac{10+1}{4}\right)^{\text{th}} \text{ or } 2.75^{\text{th}} \text{ observation.}$$

$$\therefore Q_1 = 2^{\text{nd}} \text{ observation} + 0.75 (3^{\text{rd}} \text{ observation} - 2^{\text{nd}} \text{ observation})$$

$$= 12 + 0.75 (14 - 12)$$

$$= 12 + 1.50$$

$$= 13.50 \text{ marks.}$$

$$Q_2 = \text{value of } \frac{2(n+1)}{4}^{\text{th}} \text{ observation.}$$

$$= \text{value of } \frac{2(10+1)}{4}^{\text{th}} \text{ or } 5.5^{\text{th}} \text{ observation.}$$

$$\begin{aligned}
 \therefore Q_2 &= 5^{\text{th}} \text{ observation} + 0.5 (6^{\text{th}} \text{ observation} - 5^{\text{th}} \text{ observation}) \\
 &= 22 + 0.5 (26 - 22) \\
 &= 22 + 2.0 \\
 &= 24 \text{ marks.}
 \end{aligned}$$

$$\begin{aligned}
 Q_3 &= \text{value of } \frac{3(n+1)}{4}^{\text{th}} \text{ observation.} \\
 &= \text{value of } \frac{3(10+1)}{4}^{\text{th}} \text{ or } 8.25^{\text{th}} \text{ observation}
 \end{aligned}$$

$$\begin{aligned}
 \therefore Q_3 &= 8^{\text{th}} \text{ observation} + 0.25 (9^{\text{th}} \text{ observation} - 8^{\text{th}} \text{ observation}) \\
 &= 32 + 0.25 (35 - 32) \\
 &= 32 + 0.75 \\
 &= 32.75 \text{ marks.}
 \end{aligned}$$

$$\begin{aligned}
 D_6 &= \text{value of } \frac{6(n+1)}{10}^{\text{th}} \text{ observations.} \\
 &= \text{value of } \frac{6(10+1)}{10}^{\text{th}} \text{ or } 6.6^{\text{th}} \text{ observation}
 \end{aligned}$$

$$\begin{aligned}
 D_6 &= 6^{\text{th}} \text{ observation} + 0.6 (7^{\text{th}} \text{ observation} - 6^{\text{th}} \text{ observation}) \\
 &= 26 + 0.6 (30 - 26) \\
 &= 26 + 2.4 \\
 &= 28.4 \text{ marks.}
 \end{aligned}$$

$$\begin{aligned}
 P_{75} &= \text{value of } \frac{75(n+1)}{100}^{\text{th}} \text{ observation.} \\
 &= \text{value of } 8.25^{\text{th}} \text{ observation}
 \end{aligned}$$

$$\begin{aligned}
 \therefore P_{75} &= 8^{\text{th}} \text{ observation} + 0.25 (9^{\text{th}} \text{ observation} - 8^{\text{th}} \text{ observation}) \\
 &= 32 + 0.25 (35 - 32) \\
 &= 32 + 0.75 \\
 &= 32.75 \text{ marks.}
 \end{aligned}$$

B. For simple frequency distribution or ungrouped frequency distribution

First the data are arranged in ascending order of magnitude. Then cumulative frequency (c.f.) less than type are found out. Next we have to find out the c.f. which is next higher or equal to :

(i) $\frac{k(N+1)}{4}$ for Quartiles, where

$$N = \sum f \text{ and } k = 1, 2, 3.$$

(ii) $\frac{k(N+1)}{10}$ for Decile, where

$$N = \sum f \text{ and } k = 1, 2, \dots, 9.$$

(iii) $\frac{k(N+1)}{100}$ for Percentile where

$$N = \sum f \text{ and } k = 1, 2, \dots, 99.$$

Example 28 : Find Q_1 , Q_3 , D_7 and P_{40} from the following data.

Weight (in kg):	40	42	45	50	51	54	56	59	60
No. of Persons:	2	6	8	10	6	14	12	8	14

Solution : Let us prepare the following cumulative frequency distribution table

Weight (in kg)	f	c.f.
40	2	2
42	6	8
45	8	16
50	10	26
51	6	32
54	14	46
56	12	58
59	8	66
60	14	80
	$\sum f = N = 80$	

$$Q_1 = \text{value of } \left(\frac{N+1}{4}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{80+1}{4}\right)^{\text{th}} \text{ or } 20.25^{\text{th}} \text{ observation}$$

The c.f. next higher to 20.25 is 26 which corresponds to the value 50.

$$\therefore Q_1 = 50$$

$$Q_3 = \text{value of } \frac{3(N+1)}{4} \text{th observation}$$

$$= \text{value of } \left(\frac{3 \times 81}{4}\right)^{\text{th}} \text{ or } 60.75^{\text{th}} \text{ observation}$$

The c.f. next higher to 60.75 is 66 which corresponds to the value 59

$$\therefore Q_3 = 59$$

$$D_7 = \text{value of } \frac{7(N+1)}{10} \text{th observation}$$

$$= \text{value of } \left(\frac{7 \times 81}{10}\right)^{\text{th}} \text{ or } 56.7^{\text{th}} \text{ observation}$$

The c.f. next higher to 56.7 is 58 which corresponds to the value 56.

$$\therefore D_7 = 56$$

$$P_{40} = \text{value of } \frac{40(N+1)}{100} \text{th observation}$$

$$= \text{value of } \left(\frac{40 \times 81}{100}\right)^{\text{th}} \text{ or } 32.4^{\text{th}} \text{ observation}$$

The c.f. next higher to 32.4 is 46 which corresponds to the value 54.

$$\therefore P_{40} = 54$$

C. For continuous or grouped frequency distributions :

For continuous frequency distribution :

(i) The formula for calculating quartile is

$$Q_k = L + \frac{k\left(\frac{N}{4}\right) - \text{c.f.}}{f} \times i, \quad (k = 1, 2, 3)$$

Where, L = lower limit of the quartile class

N = $\sum f$ = total frequency.

c.f. = cumulative frequency preceeding the quartile class.

f = frequency of the quartile class.

i = width of the quartile class.

(ii) Decile is determined by the formula

$$D_k = L + \frac{k\left(\frac{N}{10}\right) - \text{c.f.}}{f} i, (k = 1, 2, \dots, 9)$$

Where, L = lower limit of the decile class

N = $\sum f$ = total frequency.

c.f. = cumulative frequency preceeding the decile class.

f = frequency of the decile class.

i = width of the decile class.

(iii) Percentile determined by the formula

$$p_k = L + \frac{k\left(\frac{N}{100}\right) - \text{cf}}{f} \times i$$

(k=1, 2, ... 99)

Where, L = lower limit of percentile claks

N = $\sum f$ = total frequency

cf = cumulatime frequency of the class precuding the percatile class

f = frequency of the percentile class

i = Width of the percentile class.

Example 29 : Find Q_1 , Q_3 , D_8 and P_{65} from the following frequency distribution

Class Interval :	0–10	10–20	20–30	30–40	40–50	50–60
Frequency :	3	9	15	30	18	5

Solution : Let us prepare the following cumulative frequency distribution table.

C.I.	f	c.f.
0–10	3	3
10–20	9	12
20–30	15	27
30–40	30	57
40–50	18	75
50–60	5	80
	$\sum f = N = 80$	

$$Q_1 = \text{value of } \left(\frac{N}{4}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{80}{4}\right)^{\text{th}} \text{ or } 20^{\text{th}} \text{ observation}$$

The c.f. next higher to 20 is 27, so Q_1 lies in the class 20–30

$$\begin{aligned} \therefore Q_1 &= L + \frac{N/4 - \text{c.f.}}{f} \times i \\ &= 20 + \frac{20 - 12}{15} \times 10 \\ &= 20 + 5.33 \\ &= 25.33 \end{aligned}$$

$$Q_3 = \text{value of } \left(\frac{3N}{4}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{3 \times 80}{4}\right)^{\text{th}} \text{ or } 60^{\text{th}} \text{ observation}$$

The c.f. next higher to 60 is 75, so Q_3 lies in the class 40–50.

$$\begin{aligned} \therefore Q_3 &= L + \frac{3N/4 - \text{c.f.}}{f} \times i \\ &= 40 + \frac{60 - 57}{18} \times 10 \\ &= 40 + 1.67 \\ &= 41.67 \end{aligned}$$

$$D_8 = \text{value of } \left(\frac{8N}{10}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{8 \times 80}{10}\right)^{\text{th}} \text{ or } 64^{\text{th}} \text{ observation}$$

The c.f. next higher to 64 is 75, so D_8 lies in the class 40–50

$$\begin{aligned} \therefore D_8 &= L + \frac{8N/10 - \text{c.f.}}{f} \times i \\ &= 40 + \frac{64 - 57}{18} \times 10 \\ &= 40 + 4.11 \\ &= 44.11 \end{aligned}$$

$$P_{65} = \text{value of } \left(\frac{65N}{100}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{65 \times 80}{100}\right)^{\text{th}} \text{ or } 60^{\text{th}} \text{ observation}$$

The c.f. next higher to 52 is 57, so P_{65} lies in the class 30–40.

$$\begin{aligned} \therefore P_{65} &= L + \frac{\frac{65N}{100} - \text{c.f.}}{f} \times i \\ &= 30 + \frac{52 - 27}{30} \times 10 \\ &= 30 + 8.33 \\ &= 38.33 \end{aligned}$$

4.10.2 MERITS, DEMERITS AND USES OF QUARTILES, DECILES AND PERCENTILES

Merits :

- (i) They can be computed for distributions with open-end class interval.
- (ii) They are not affected by extreme values.
- (iii) They can be obtained graphically.

Demerits :

- (i) While computing the data have to be arranged in ascending order.
- (ii) They do not depend on all the observations.

Uses :

- (i) In data relating to business and economic statistics, quartiles are useful in personal work and productivity ratings.
- (ii) Percentiles are useful in scaling and ranking of test scores in psychological and educational statistics.

Note 1 : If quartile, decile or percentile lies in the first class interval then the c.f. of the preceding class-interval shall be taken as zero.

Note 2 : The relations among quartiles, deciles and percentiles are as follows.

$$Q_1 = P_{25}, Q_2 = D_5 = P_{50}, Q_3 = P_{75}$$

$$D_1 = P_{10}, D_2 = P_{20}, \dots, D_9 = P_{90}$$

4.10.3 GRAPHICAL METHOD OF DETERMINING QUARTILES, DECILES AND PERCENTILES :

Quartiles, Deciles and Percentiles can be determined graphically with the help of cumulative frequency curve or ogive. The procedure is same as that of median. Draw an ogive by “less than” method. Then determine the quartiles, deciles and percentiles by the formula—

$$\frac{KN}{4}, (k = 1, 2, 3), \frac{KN}{10}, (k = 1, 2, \dots, 9)$$

and $\frac{KN}{100}, (k = 1, 2, \dots, 99)$ respectively.

Now, locate these values on the y-axis and from them draw perpendiculars on the ogive. From the points where they meet the ogive draw perpendiculars on the x-axis and the points where they meet x-axis are the required quartiles, deciles and percentiles respectively.

Example 30 : Find Q_1 and Q_3 graphically from the data given in example.

Solution : After finding c.f. and preparing the table as in example we get

$$Q_1 = \text{value of } \left(\frac{N}{4}\right)^{\text{th}} \text{ observation}$$

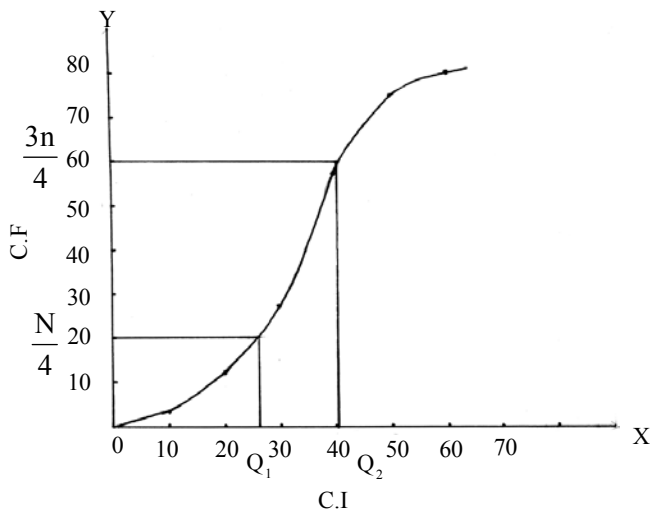
$$= \text{value of } \left(\frac{80}{4}\right)^{\text{th}} \text{ or } 20^{\text{th}} \text{ observation and}$$

$$Q_3 = \text{value of } \left(\frac{3N}{4}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{3 \times 80}{4}\right)^{\text{th}} \text{ or } 60^{\text{th}} \text{ observation}$$

Now draw an ogive by the “less than” method. Taking 20 and 60 on the y-axis draw perpendiculars on the ogive. From the points where they meet the ogive draw perpendiculars on the x-axis. The points where they meet the x-axis are the Q_1 & Q_3 respectively.

It is clear from the graph that the value of Q_1 is 25.33 (approx) and the value of Q_3 is 41.67 (approx).



EXERCISE (4E)

1. Define median and discuss its relative merits and demerits.
2. Describe the similarities and differences among median, quartiles, deciles and percentiles.
3. Find the median from the following data

(i) 70, 60, 75, 90, 65, 80, 42, 65, 72 [Ans: 70]

(ii) 6, 10, 4, 3, 9, 11, 22, 18 [Ans: 9.5]

4. Find the median from the following frequency distribution.

Marks :	20	9	25	50	40	80
No. of Students :	6	4	16	7	8	2

[Ans : 25]

5. The following table gives the marks obtained by 100 students. Find the median mark.

Marks :	0–10	10–20	20–30	30–40	40–50
No. of Students :	14	23	27	21	15

[Ans : 24.81]

6. Find the missing frequencies from the following frequency distribution. The median of the distribution is 30.

C.I. :	0–10	10–20	20–30	30–40	40–50	50–60
f:	10	?	25	30	?	10

[Ans : 15 & 10]

7. Calculate the mean and median from the following data.

Class Interval :	1–10	11–20	21–30	31–40	41–50	51–60	61–70
Frequency :	8	15	25	20	16	10	6

[Ans: mean = 33, median = 31.5]

8. From the following scores of 12 students in an examination find Q_1 , Q_3 , D_7 and P_{33} .

15, 9, 18, 20, 21, 26, 14, 13, 27, 22, 16, 8

[Ans: $Q_1=14.25$, $Q_3=25$, $D_7=22.4$, $P_{33}=15.29$]

9. Find Q_1 , Q_3 , D_4 and P_{60} from the following data.

x :	5	14	16	22	27	55	70	85	92	100
f:	8	6	14	12	10	5	16	7	4	2

[Ans: $Q_1 = 16$, $Q_3 = 70$, $D_4 = 22$, $P_{60} = 55$]

10. Determine the value of median from the following data.

Income (Rs.):	Below 1000	1000–2000	2000–3000	3000–4000	above 4000
No. of persons :	4	6	10	7	3

[Ans: Rs.2500]

11. Calculate Median, Q_1 , Q_3 , D_8 and P_{30} from the following distribution.

Wages :	150–160	160–170	170–180	180–190	190–200
(in Rs.)					

No. of workers :	62	65	71	83	90
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Wages (Rs.):	200–210	210–220	220–230	230–240	240–250
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No. of workers :	95	110	80	68	61
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[Ans: Median = 202.26, $Q_1 = 179.75$, $Q_3 = 221.59$, $D_8 = 26.5$, $P_{30} = 184.51$]

12. Draw an ogive and locate median, first quartile and third quartile from the following data.

Marks :	0–10	10–20	20–30	30–40	40–50	50–60
No. of Students :	3	9	15	30	18	5

[Ans: Median = 34.33, $Q_1 = 25.33$, $Q_3 = 41.67$]

4.11 MODE

Mode is defined as that value in a series which occurs maximum number of times. In a frequency distribution mode is that variate which has the maximum frequency. For eg, in a series 4, 5, 6, 3, 2, 1, 6, 7, 8, 4, 6, 2, we notice that 6 occurs maximum number of times, therefore 6 is the mode.

A series of observations may have one or more modes. The series with one mode is called unimodal. In a series, the mode is ill-defined if it contains more than one mode. A series having two modes is called bimodal and if it has three or more modes is called trimodal or multimodal.

4.11.1 CALCULATION OF MODE

A. For individual series

In case of individual series, the value which is repeated maximum number of times is the mode of the series.

B. For simple frequency distributions (Ungrouped data)

In case of frequency distribution, mode is the value of the variable corresponding to the maximum frequency.

Example 31 : Find out the modal family size from the following data

Family size (X) :	2	3	4	5	6	7
No. of families (F) :	70	82	93	110	88	74

Solution : Let us prepare the table showing the frequency as follows :

Family size (x)	No. of families (f)
2	70
3	82
4	93
5	110
6	88
7	74

In the above table we notice that the size 5 occurs maximum number of times. It has the highest frequency size 110.

\therefore 5 is the mode of the distribution.

C. Calculation of mode of grouping method :

The value of mode is determined by the method of grouping if :

- (i) The maximum frequency is repeated.
- (ii) The maximum frequency occurs in the very beginning or at the end of the distribution.
- (iii) There are irregularities in the distribution.

To calculate mode by the method of grouping first we prepare the grouping table as follows :

The frequencies in column (1) are the original frequencies. Column (2) is obtained by combining the frequencies two by two. If we leave the first frequency and combine the remaining frequencies two by two we get column (3). We proceed to combine the original frequencies or frequencies in column (1) three by three to get column (4). The combination of frequencies three by three after leaving the first frequency of column (1) we get the column (5). After leaving the first two frequencies of column (1) and combining the frequencies three by three we get column (6). The maximum frequency in each column is circled.

After preparing the grouping table, we prepare the analysis table. While preparing this table we put the column number in first column or column (i), then the maximum frequency in the second column or column (ii). The value or combination of values of x having maximum frequency indicated in column (ii) in the third column or column (iii).

The procedure will be clear from the following example.

Example 32 : Determine the mode of the following data.

Marks :	20	21	22	23	24	25	26	27	28	29
No. of Students :	6	9	4	2	10	8	7	5	1	1

Solution : Here we observe that the distribution is not regular. Here we cannot say just by inspection that maximum frequency is 10 and mode is 24. Here we shall calculate mode by the method of grouping.

Grouping Table :

Size (x)	Frequency					
	1	2	3	4	5	6
20	6	}15	}13	}15	}15	}16
21	9					
22	4	}6		}12		
23	2					
24	10	}18	}15	}25		
25	8					
26	7	}12	}13	}9		
27	5					
28	1	}4				
29	3					

Analysis table :

Col (i) Col ⁿ number	Col (ii) Max Frequency	Col (iii) Value or combination of values of x giving max freq. in Col (ii)
1	10	24
2	18	24, 25
3	15	25, 26
4	20	23, 24, 25
5	25	24, 25, 26
6	20	25, 26, 27

Here in Col (iii), we find that 25 is repeated maximum number of times and hence the value of mode is 25.

D. Calculation of mode for a continuous or grouped frequency distribution.

In a continuous or grouped frequency distribution. To find mode we have to determine the modal class ie, the class corresponding to the maximum frequency in a distribution. The modal class can be determined either by inspection or with the help of grouping table. After finding the modal class, we calculate mode by applying the following formula.

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

Where, L = lower limit of the modal class.

f_1 = frequency of the modal class.

f_0 = frequency of the class preceeding the modal class.

f_2 = frequency of the class succeeding the modal class.

i = width of the modal class.

Example 33 :

Calculate mode for the following frequency distribution.

Income per day :	0–10	10–20	20–30	30–40	40–50
(in Rs.)					
No. of Persons :	10	14	19	17	13

Solution : For calculation let us prepare the following table.

Income (in Rs.) (C.I.)	No. of Persons (f)
0–10	10
10–20	14 → f_0
20–30	19 → f_1
30–40	17 → f_2
40–50	13

From the table it is clear that the maximum frequency is 19 and it lies in the class 20–30. Therefore the modal class is 20–30.

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

Here $L = 20$, $f_0 = 14$, $f_1 = 19$, $f_2 = 17$ & $i = 10$

$$\begin{aligned} \therefore \text{Mode} &= 20 + \frac{19 - 14}{2 \times 19 - 14 - 17} \times 10 \\ &= 20 + \frac{5}{7} \times 10 \\ &= 20 + 7.143 \\ &= \text{Rs. } 27.14 \end{aligned}$$

Remarks :

1. Mode requires the distribution to be continuous with exclusive class interval.
2. In case of cumulative frequency distribution, it must be converted to simple frequency distribution before finding mode.
3. In case of open-end class-interval, it is required to close the ends and complete the class intervals.

4.11.2 MERITS, DEMERITS AND USES OF MODE

Merits

1. It is easy to understand and calculate.
2. It is not at all affected by extreme observations.
3. Mode can be estimated graphically from a histogram.
4. In some cases, it can be located by inspection.

Demerits

1. It is not based on all the observations of the series.
2. Mode is not suitable for further mathematical treatment.
3. Mode is ill-defined if the maximum frequency is repeated.
4. As compared with mean, mode is affected to a greater extent by the fluctuations of sampling.

Uses

Mode is of great use to large scale manufactures of readymade garments, shoes and so on. In all such cases, it is important to know the size that fits most persons rather than the mean size.

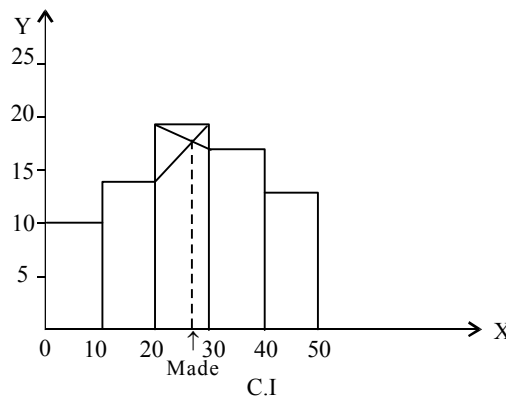
4.11.3 GRAPHICAL METHOD OF DETERMINING MODE

In case of frequency distribution, mode can be located graphically as follows:

- ♦ A histogram is drawn from the given data, the tallest rectangle will represent the modal class.
- ♦ Two diagonal lines are drawn from the top right corner and left corner of the tallest rectangle to the top right and left corner of the adjacent rectangles.
- ♦ A perpendicular line is drawn from the point of intersection of the two diagonal lines on the x-axis. The point where the perpendicular meets the x-axis is the value of mode.

Example 34 : Estimate mode graphically from the data given in example 33.

Solution : Let us construct a histogram from the given data.



From the graph the mode is found to be 27 (approx).

We have observed that the actual calculation of mode was 27.14. Thus we see that the estimation of mode by graphic method and by calculation is more or less same.

4.11.4 RELATIONSHIP BETWEEN MEAN, MEDIAN AND MODE

In a symmetrical distribution mean, median and mode are equal. In a modality skewed or asymmetrical distribution, the relationship between Mean, Median and Mode is given by Karl Pearson as follows :

$$\text{Mean} - \text{Mode} = 3 (\text{Mean} - \text{Median})$$

$$\Rightarrow \text{Mode} = 3 \text{ Median} - 2 \text{ Mean}$$

The above relationship is known as Empirical relationship.

Example 35 :

In a moderately skewed distribution the values of median and mean are 72 and 78 respectively, find the value of mode.

Solution : From the empirical relation we have

$$\text{Mean} - \text{Mode} = 3 (\text{Mean} - \text{Median})$$

$$\Rightarrow \text{Mode} = 3 \text{ Median} - 2 \text{ Mean}$$

$$= 3 \times 72 - 2 \times 78$$

$$= 216 - 156$$

$$= 60 \text{ Ans.}$$

Example 36 : Compute Mean, Median and Mode from the following frequency distributions.

Wages (in Rs.) :	0–100	100–200	200–300	300–400	400–500
No. of Persons :	3	7	10	6	2

Solution : Let us prepare the following table for calculation of mean, median and mode.

Wages (in Rs.) C.I.	Mid value (x)	f	$d' = \frac{X - A}{i}$	fd'	c.f.
0–100	50	3	-2	-6	3
100–200	150	7	-1	-7	10
200–300	250	10	0	0	20
300–400	350	6	1	6	26
400–500	450	2	2	4	28
Total		$\Sigma f = N$ = 28		$\Sigma fd' = -3$	

Let the assumed mean $A = 250$

$$\begin{aligned}\text{Mean} = \bar{x} &= A + \frac{\sum fd'}{N} \times i \\ &= 250 + \frac{(-3)}{28} \times 100 \\ &= 250 - 10.71 \\ &= 239.29\end{aligned}$$

\therefore Mean wage = Rs. 239.29

$$\text{Here } \frac{N}{2} = \frac{28}{2} = 14$$

The c.f. just greater than 14 is 20, therefore the corresponding class 200–300 is the median class

$$\text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Here $L = 200$, $\frac{N}{2} = 14$, c.f. = 10, $f = 10$ & $i = 100$

$$\begin{aligned}\text{Median} &= 200 + \frac{14 - 10}{10} \times 100 \\ &= 200 + 40 = 240\end{aligned}$$

\therefore Median = Rs. 240

By inspection, we observe that modal class is 200–300

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

Here $L = 200$, $f_1 = 10$, $f_0 = 7$, $f_2 = 6$ & $i = 100$

$$\begin{aligned}\therefore \text{Mode} &= 200 + \frac{10 - 7}{20 - 7 - 6} \times 100 \\ &= 200 + \frac{3}{7} \times 100 \\ &= 200 + 42.86 \\ &= 242.86\end{aligned}$$

\therefore Mode = Rs. 242.86

EXERCISE (4F)

1. Define mode. Write down the merits and demerits of mode.
2. What do understand by unimodal and bi-modal distributions.
3. Find the mode of 8, 7, 6, 8, 8, 5, 6.

[Ans: 8]

4. Find the mode of variable x.

x :	1	2	3	4	5	6	7	8	9
f :	3	4	18	25	40	30	22	10	6

[Ans: 5]

5. Write down the empirical relation between mean, median and mode.
6. Find mode for the following frequency distribution.

Size of shoes :	1	2	3	4	5	6	7	8	9	10	11	12
No. of Pairs :	3	8	15	23	35	40	32	28	20	45	14	6

[Hint: Prepare grouping & analysis table]

[Ans: 6]

7. Compute mode for the following frequency distribution.

Production per day : (in tons)	20–22	22–24	24–26	26–28	28–30
No. of days :	7	13	22	10	8

[Ans: 24.86 tons]

8. Find the mode of family expenditure given below.

Family expenditure : (in Rs.)	40–59	60–79	80–99	100–119	120–139
No. of family :	25	130	250	70	25

[Ans: Rs. 87.50]

9. Find the mode from the following data.

Marks :	less than 5,	5–10	10–15	15–20	more than or equal to 20
No. of Students :	4	15	8	5	2

[Ans: 8.06 marks]

10. Calculate the mode for the following data

Marks	No. of students
less than	10 5
” ”	20 9
” ”	30 15
” ”	40 18
” ”	50 20

[Ans: 24 marks]

11. In a moderately asymmetrical distribution Mode and Mean are 32.1 and 35.4 respectively. Find the median.

[Ans: 34.3]

12. Calculate Mean and Median of the following frequency distribution given below. Hence calculate Mode using the empirical relation between the three.

Marks :	0–10	10–20	20–30	30–40	40–50	50–60
No. of students :	4	6	20	10	7	3

[Ans: Mean = 28.8, Median = 27.5 & Mode = 24.9]

13. Represent the following data by a histogram and find out the modal uses.

Class Interval :	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40
Frequency :	4	6	10	10	25	22	18	5

[Ans: 24 (approx)]

14. State whether the statements are true or false.

- (i) AM is affected by extreme observations.
- (ii) GM is used in the construction of Index numbers.
- (iii) HM can be computed when there are negative observations.
- (iv) A distribution may have more than one mode.
- (v) Median is a positional measure of average.
- (vi) AM of a set of values lie between GM and HM.

[Ans: (i) True, (ii) True, (iii) False,
(iv) True, (v) True, (vi) False]

15. Fill in the blanks :

- (i) _____ is not affected by extreme observations.
- (ii) _____ is based on all the observations.
- (iii) Average suited for qualitative phenomena is _____.
- (iv) A distribution with two modes is called _____.
- (v) _____ = $\sqrt{AM \times HM}$
- (vi) If a constant k is added to each of the observation, then AM will _____ by k.

[Ans: (i) Median, (ii) Mean, (iii) Median,
(iv) Bimodal, (v) G.M., (vi) Increased]

UNIT-5**MEASURES OF DISPERSION
(VARIATION)****OBJECTIVES :**

This chapter will help you to understand–

- ➔ The meaning, purpose and characteristics of dispersion.
- ➔ The different measures of dispersion, their merits, demerits and uses.
- ➔ The calculation of dispersion for individual series, ungrouped and grouped frequency distributions.
- ➔ The meaning and calculation of coefficient of variation.

5.1 INTRODUCTION

Statistical averages fail to reveal the degree or extent of variability of individual items of a distribution from the central value. For instance, let us consider the following three series :

Series	Observations	Total	Mean
I	10, 10, 10, 10, 10	50	10
II	8, 9, 10, 11, 12	50	10
III	2, 6, 10, 14, 18	50	10

All the above three series have the same number of observations ($n = 5$) and the same mean ($\bar{x} = 10$). Thus if we are given that the mean of a series of 5 observations is 10, we will not be able to say whether it is the average of the I or II or III series. Hence statistical averages or measures of central tendency are inadequate in giving us a complete idea of the distribution. So they must be supported and supplemented by some other measures. One such measure is that of dispersion.

The literal meaning of dispersion is ‘scatteredness’. We study dispersion to have an idea of the homogeneity or heterogeneity of a distribution. Dispersion measures the degree to which data tend to spread around an average.

According to L.R. Connor, “Dispersion is a measure of the extent to which the individual items vary.”

5.2 CHARACTERISTICS OR REQUIREMENTS OF A GOOD MEASURE OF DISPERSION :

To be an ideal measure of dispersion, it must possess the following characteristics :

- (i) It should be rigidly defined.
- (ii) It should be easy to understand and calculate.
- (iii) It should be based on all the observations of the series.
- (iv) It should be affected as little as possible by fluctuations of sampling.
- (v) It should be capable of further algebraic treatment.
- (vi) It should not be unduly affected by extreme observations.

[The above characteristics are same as those for an ideal measure of central tendency]

5.3 ABSOLUTE AND RELATIVE MEASURES OF DISPERSION :

Absolute Measures : The measures of dispersion which are expressed in terms of original units of a distribution are called absolute measures. For example, if we calculate dispersion of a series, say monthly income earned by the employees of a company, then the calculated dispersion will also be in the same unit ie, Rs. The absolute measures are not used for comparing the variability of two or more distributions.

Relative Measures : These are the measures of dispersion which are independent of the original units of the variables. We use these measures for comparing the variability of two or more series. Relative measures are generally in the form of pure number, coefficients, percentages etc.

5.4 DIFFERENT MEASURES OF DISPERSION :

The different absolute measures of dispersion are :

- (i) Range.
- (ii) Quartile Deviation (Q.D.) or semi Interquartile Range.
- (iii) Mean Deviation (M.D.).
- (iv) Standard Deviation (S.D.).
- (v) Lorenz curve – is a graphic method of studying variability.

Their corresponding relative measures are –

- (i) Coefficient of Range.
- (ii) Coefficient of Q.D.
- (iii) Coefficient of H.D.
- (iv) Coefficient of S.D.

5.5 RANGE :

Range is defined as the difference between the maximum and minimum values of the observations of the distribution.

$$\therefore \text{Range} = X_{(\max)} - X_{(\min)}$$

where X denotes an observation of the distribution.

The relative measure of dispersion corresponding to range is coefficient of range.

Coefficient of range is defined as the ratio of the difference between the two extreme observations of the distribution to their sum.

\therefore Coefficient of range

$$= \frac{X_{\max} - X_{\min}}{X_{\max} + X_{\min}}$$

5.5.1 CALCULATION OF RANGE :**A. Calculation of range for individual series :**

$$\text{Range} = X_{\max} - X_{\min}$$

Example 1 : Find the range and coefficient of range for the following data given in Kg.

65, 70, 40, 45, 55, 30

Here the maximum observation is 70 kg and the minimum observation is 30 kg.

$$\therefore \text{Range} = 70 \text{ kg} - 30 \text{ kg} = 40 \text{ kg}$$

$$\text{Coefficient of range} = \frac{70 - 30}{70 + 30} = \frac{40}{100} = 0.4$$

B. Calculation of range for ungrouped frequency distribution :

In case of frequency distributions, the frequencies of the various variate values are immaterial as the range depends only on the two extreme observations.

Example 2 : Find the range and its coefficient from the following data.

Wages/day :	170	175	180	185	190	200
(in Rs.)						
No. of Persons :	4	16	10	6	4	5

Solution : Range = 200–170 = Rs. 30.

$$\text{Coefficient of range} = \frac{200 - 170}{200 + 170} = \frac{20}{370} = 0.081$$

C. Calculation of range and its coefficient for the following frequency distribution.

In case of grouped frequency distribution for discrete values or continuous frequency distribution, range is defined as the difference between the upper limit of the highest class and lower limit of the smallest class.

Example 3 : Find the range and its coefficient for the following frequency distribution.

Marks :	10–20,	20–30,	30–40,	40–50
No. of Students :	15	20	16	8

Solution : Range = 50–10 = 40 marks.

$$\text{Coefficient of range} = \frac{50 - 10}{50 + 10} = \frac{40}{60} = 0.667$$

Note : In case of inclusive class interval it must be converted to exclusive class interval before finding range to maintain continuity.

5.5.2 MERITS, DEMERITS AND USES OF RANGE :

Merits

1. It is rigidly defined.
2. It is easy to calculate and understand.

Demerits

1. It is not based on all the observations of a series.
2. Range cannot be computed for open end classes.
3. It is very much affected by fluctuations of sampling.
4. It is not capable of further algebraic treatment.
5. It is highly affected by extremes observations.

Uses

1. It is used in industry for statistical quality control.
2. It is used in stock markets to study the fluctuations.
3. It is used by the meteorological department for weather forecasts.

EXERCISE (5A)

1. What do you mean by dispersion?
2. Define measures of dispersion.
3. Write down the characteristics of an ideal measure of dispersion.
4. Define absolute and relative measure of dispersion.
5. Define range. Write down the merits, demerits and uses of range.
6. Find range and coefficient of range for the following data : Rs. (20, 21, 22, 25, 30, 45, 50, 60).

[Ans: Rs.40, 0.5]

7. Find the range and coefficient of range from the following data :

Marks :	8	18	22	25	29	37	48
No. of Students :	1	5	10	4	12	6	1

[Ans: 40 marks, 0.71]

8. Find the range and its coefficient for the following data :

Age :	16–20	21–25	26–30	31–35
No. of Persons :	10	15	17	8

[Ans: 20 yrs, 0.39]

5.6 QUARTILE DEVIATION (Q.D.) :

Quartile deviation is defined as the difference between upper quartile and lower quartile divided by two.

$$\text{ie, Q.D.} = \frac{Q_3 - Q_1}{2}$$

Quartile deviation is also known as semi-interquartile range. It is an absolute measure of dispersion. The relative measure ie, coefficient of quartile deviation is given by

$$\begin{aligned} \text{Coefficient of Q.D.} &= \frac{\frac{Q_3 - Q_1}{2}}{\frac{Q_3 + Q_1}{2}} \\ &= \frac{Q_3 - Q_1}{Q_3 + Q_1} \end{aligned}$$

5.6.1 CALCULATION OF QUARTILE DEVIATION :**(A) For individual series :**

In case of individual series we arrange the data in ascending order to find quartile deviation.

Example 4 : Find Q.D. and coefficient of Q.D. of the following observations.

5, 8, 7, 10, 15, 12, 14

Solution : Let us arrange the data in ascending order of magnitude

5, 7, 8, 10, 12, 14, 15

Here $n = 7$

$$Q_1 = \text{value of } \frac{(n+1)^{\text{th}}}{4} \text{ observation}$$

$$= \text{value of } \left(\frac{7+1}{4}\right)^{\text{th}} \text{ on 2}^{\text{nd}} \text{ observation}$$

$$\therefore Q_1 = 7$$

$$Q_3 = \text{value of } \frac{3(n+1)^{\text{th}}}{4} \text{ observation}$$

$$= \text{value of } \frac{3(7+1)^{\text{th}}}{4} \text{ on 6}^{\text{th}} \text{ observation}$$

$$\therefore Q_3 = 14$$

$$\therefore \text{Q.D.} = \frac{Q_3 - Q_1}{2} = \frac{14 - 7}{2} = 3.5$$

$$\therefore \text{Coefficient of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{14 - 7}{14 + 7}$$

$$= \frac{7}{21}$$

$$= 0.33$$

B. Calculation of Q.D. for ungrouped frequency distribution :

To compute Q.D. for ungrouped frequency distribution we have to find the cumulative frequency 'less than' type.

Example 5 : Compute Q.D. and its coefficient for following data :

x :	10	12	15	20	30	35	50	55
y :	4	6	8	12	16	3	5	6

Solution : Let us prepare the following table

x	f	c.f.
10	4	4
12	6	10
15	8	18
20	12	30
30	16	46
35	3	49
50	5	54
55	6	60
	$\Sigma f = N = 60$	

$$Q_1 = \text{value of } \left(\frac{N+1}{4}\right)^{\text{th}} \text{ observation}$$

$$= \text{value of } \left(\frac{60+1}{4}\right)^{\text{th}} \text{ on } 15.25^{\text{th}} \text{ observation}$$

The c.f. just greater than 15.25 is 18

$$\therefore Q_1 = 15$$

$$Q_3 = \text{value of } \frac{3(n+1)}{4}^{\text{th}} \text{ observation}$$

$$= \text{value of } \frac{3(60+1)}{4}^{\text{th}} \text{ on } 45.75^{\text{th}} \text{ observation}$$

The c.f. just greater than 45.75 is 46

$$\therefore Q_3 = 30$$

$$\text{Now Q.D.} = \frac{Q_3 - Q_1}{2}$$

$$\therefore \text{Q.D.} = \frac{30 - 15}{2} = 7.5$$

$$\begin{aligned} \therefore \text{Coefficient of Q.D.} &= \frac{Q_3 - Q_1}{Q_3 + Q_1} \\ &= \frac{30 - 15}{30 + 15} = \frac{15}{45} \\ &= 0.33 \end{aligned}$$

C. Calculation of Q.D. for grouped or continuous frequency distribution

To calculate Q.D. for grouped or continuous frequency distribution we find the cumulative frequency 'less than' type and find the 1st and 3rd quartiles.

Example 6 : Find the Q.D. and its coefficient from the following data.

Wages per hour : (in Rs.)	25–35	35–45	45–55	55–65	65–75
Labourers :	2	10	25	16	7

Solution : Let us prepare the following table

Wages per hour (in Rs.) x	Labourers f	c.f.
25–35	2	2
35–45	10	12
45–55	25	37
55–65	16	53
65–75	7	60
	$\Sigma f = N = 60$	

Here, $Q_1 =$ value of $\left(\frac{N}{4}\right)^{\text{th}}$ observation

$=$ value of $\frac{60}{4}$ or 15th observation.

The c.f. just greater than 15 is 37, which corresponds to the class 45–55.

$\therefore Q_1$ lies in the class 45–55.

$$Q_3 = L + \frac{\frac{3N}{4} - \text{c.f.}}{f} \times i$$

Here, $L=45$, $\frac{N}{4}=15$, $\text{c.f.}=12$, $f=25$ & $i=10$

$$\begin{aligned} \therefore Q_1 &= 45 + \frac{15 - 12}{25} \times 10 \\ &= 45 + 1.2 = 46.2 \end{aligned}$$

$Q_3 =$ value of $\left(\frac{3N}{4}\right)^{\text{th}}$ observation

$=$ value of $\left(\frac{3 \times 60}{4}\right)^{\text{th}}$ on 45th observation

The c.f. just greater than 45 is 53, which corresponds to the class 55–65

$\therefore Q_3$ lies in the class 55–65.

$$Q_3 = L + \frac{\frac{N}{4} - \text{c.f.}}{f} \times i$$

Here, $L=55$, $\frac{3N}{4}=45$, $\text{c.f.}=37$, $f=16$ & $i=10$

$$\begin{aligned} \therefore Q_3 &= 55 + \frac{45 - 37}{16} \times 10 \\ &= 55 + 5 = 60 \end{aligned}$$

$$\begin{aligned} \therefore \text{Q.D.} &= \frac{Q_3 - Q_1}{2} = \frac{60 - 46.2}{2} = \frac{13.8}{2} \\ &= \text{Rs. } 6.9 \end{aligned}$$

$$\begin{aligned} \text{Coefficient of Q.D.} &= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{60 - 46.2}{60 + 46.2} \\ &= \frac{13.8}{106.2} \\ &= 0.129 \end{aligned}$$

5.6.2 MERITS, DEMERITS AND USES OF Q.D.

Merits

1. Q.D. is easy to understand and calculate.
2. Q.D. is based on 50% of the data, while range is based on only two observation.
3. Q.D. can be computed for frequency distribution with open end classes.
4. It is not at all affected by extreme observations.

Demerits

1. It is not based on all the observations of a series. Q.D. ignores 25% of the data at the both upper and lower end of the distribution.
2. Q.D. is not suitable for further mathematical treatment.
3. Q.D. is affected considerable by fluctuations of sampling.

Uses

1. Q.D. is use when class-intervals are open or when extreme observations are present in a distribution.
2. It is also used in description statistics.

EXERCISE (5 B)

1. Define Quartile deviation. What are its merits and demerits?
2. Find the Q.D. and its coefficient from the following data :
 - (i) 7, 10, 8, 6, 12, 15, 14 [Ans: 3.5, 0.33]
 - (ii) 69, 75, 72, 73, 70, 71, 74, 76, 74, 70 [Ans: 2.5, 0.034]
3. Compute Q.D. from the following data. Also find its coefficient

Height :	58	59	60	61	62	63	64	65	66
(in inches)									
No. of Students :	15	20	32	35	33	22	20	10	8

[Ans: 1.5 inches, 0.024]
4. Calculate the Q.D. and its coefficient for wages earned per hour by 86 labourers.

Wages :	30–32	32–34	34–36	36–38	38–40	40–42	42–44
(in Rs.)							
Labourers :	12	18	16	14	12	8	6

[Ans: Rs.2.85, 0.079]
5. Calculate an appropriate measure of dispersion for the following data :

Marks :	less than 36	35–37	38–40	41–43	above 43
No. of Students :	14	62	99	18	7

[Ans: Rs.1.75]
6. Find out the semi interquartile range for the following distribution :

Wages (in Rs.)	No. of workers
less than 10	5
less than 20	8
less than 30	15
less than 40	20
less than 50	30
less than 60	33
less than 70	35

[Ans: Rs. 12.59]

7. From the following data calculate the percentage of workers getting wages :

(i) more than 44

(ii) between Rs.22 and Rs.58

(iii) the quartile deviation

Wages (in Rs.) : 0–10 10–20 20–30 30–40 40–50 50–60 60–70 70–80

No. of workers : 20 45 85 160 70 55 35 30

[Ans: (i) 32.4%, (ii) 68.4%, (iii) Rs.11.12]

5.7 MEAN DEVIATION OF AVERAGE DEVIATION (M.D.)

Mean deviation (MD) is the average of the deviations of observations from any measure of central tendency. The deviations are taken as positive, ignoring the negative sign.

Thus if x_1, x_2, \dots, x_n are 'n' given observations then M.D. about an average 'A' is given by–

M.D. (about an average A) $= \frac{1}{n} \sum |x - A|$. Where $|x - A|$ is the modulus value or absolute value of the deviations after ignoring the negative sign.

We can take mean deviation from mean, median or mode. Thus we have

$$\text{MD (about mean)} = \frac{1}{n} \sum |x - \bar{x}|$$

$$\text{MD (about median)} = \frac{1}{n} \sum |x - \text{Median}|$$

$$\text{MD (about mode)} = \frac{1}{n} \sum |x - \text{Mode}|$$

The relative measure ie, coefficient of M.D. is given by

$$\text{Coefficient of MD} = \frac{\text{MD}}{\text{Average about which it is calculated}}$$

$$\text{Thus, Coefficient MD, about mean} = \frac{\text{MD}}{\text{Mean}}$$

$$\text{Coefficient of MD, about Median} = \frac{\text{MD}}{\text{Median}}$$

$$\text{Coefficient of MD, about Mode} = \frac{\text{MD}}{\text{Mode}}$$

Note :

In practice MD, about median, is preferred than M.D., about mean, since the sum of deviation calculated from median is usually less than that calculated from mean. Mode is not considered as it may be ill defined. Therefore M.D. is calculated either about mean or median.

5.7.1 CALCULATION OF MEAN DEVIATION (M.D.) :**A. Calculation of M.D. for individual series :**

We have to find the average about which it is to be calculated. Then we must take the deviations ignoring the negative signs. Next the total deviations must be divided by the number of observations.

Thus, M.D. (about an average A) = $\frac{\sum |x - A|}{n}$ where, $|x - A|$ is the absolute value (ignoring the negative sign) of the deviation of the given values of x from the average A.

Example 7 : Calculate MD about mean and median for 25, 28, 32, 45, 60.

Solution : For calculation of M.D. we prepare the following table by arranging the data in ascending order of magnitude.

x	$ x - \bar{x} = x - 38 $	$ x - \text{Median} = x - 32 $
25	13	7
28	10	4
32	6	0
45	7	13
60	22	28
	$\sum x - \bar{x} = 58$	$\sum x - \text{Median} = 52$

$$\begin{aligned} \text{Mean } (\bar{x}) &= \frac{25 + 28 + 32 + 45 + 60}{5} \\ &= \frac{190}{5} \\ &= 38 \end{aligned}$$

$$\begin{aligned} \text{M.D. about mean} &= \frac{\sum |x - \bar{x}|}{n} \\ &= \frac{58}{5} = 11.6 \end{aligned}$$

To calculate median we arrange the data in ascending order of magnitude ie, 25, 28, 32, 45, 60

$$\begin{aligned}\text{Median} &= \text{value of } \left(\frac{n+1}{2}\right)^{\text{th}} \text{ observation} \\ &= \text{value of } \left(\frac{5+1}{2}\right)^{\text{th}} \text{ or } 3^{\text{rd}} \text{ observation}\end{aligned}$$

$$\therefore \text{Median} = 32$$

$$\begin{aligned}\therefore \text{M.D. about median} &= \frac{\sum |x - \text{Median}|}{n} \\ &= \frac{52}{5} \\ &= 10.4\end{aligned}$$

B. Calculation of M.D. for ungrouped frequency distribution :

In case of ungrouped frequency distribution M.D. about an average A is given by–

$$\text{M.D. (about an average A)} = \frac{\sum f|x-A|}{N}$$

where x is the value of the variable and $f|x-A|$ is the absolute value of the deviation of the given value of x from the average 'A' multiplied by the corresponding frequency 'f', $N = \sum f$.

Example 8 : Find the mean deviation from (i) mean and (ii) median for the following data and also find its coefficients.

Marks :	4	6	8	10	12
No. of Students :	1	4	6	4	1

Solution :

(i) For calculation of MD from mean we prepare the following table.

Marks x	No. of Students f	fx	$ x - \bar{x} = x - 8 $	f x-8
4	1	4	4	4
6	4	24	2	8
8	6	48	0	0
10	4	40	2	8
12	1	12	4	4
	$\sum f = N = 16$	$\sum fx = 128$		$\sum f x-8 = 24$

$$\bar{x} = \frac{\sum fx}{N} = \frac{128}{16} = 8$$

$$\text{M.D. (about mean)} = \frac{\sum f|x - \bar{x}|}{N}$$

$$= \frac{24}{16}$$

$$= 1.5 \text{ marks}$$

Coefficient of MD (about mean)

$$= \frac{\text{MD}}{\text{Mean}}$$

$$= \frac{1.5}{8}$$

$$= 0.19$$

(ii) For calculation of MD from median we prepare the following table.

Marks x	No. of Students f	c.f.	x-Median = x-8	f x-Median
4	1	1	4	4
6	4	5	2	8
8	6	11	0	0
10	4	15	2	8
12	1	16	4	4
	$\sum f = N = 16$			$\sum f x - \text{Median} = 24$

Median = value of $\left(\frac{N+1}{2}\right)^{\text{th}}$ observation

= value of $\left(\frac{16+1}{2}\right)^{\text{th}}$ or 8.5th observation

The c.f. just greater than 8.5 is 11 which corresponds to the value 8

\therefore Median = 8

$$\text{M.D. (about median)} = \frac{\sum f|x - \text{Median}|}{N}$$

$$= \frac{24}{16}$$

$$= 1.5 \text{ marks}$$

Coefficient of MD (about median)

$$\begin{aligned} &= \frac{\text{MD}}{\text{Median}} \\ &= \frac{1.5}{8} \\ &= 0.19 \end{aligned}$$

C. Calculation of M.D. for grouped or continuous frequency distribution :

For grouped or continuous frequency distribution MD about an average 'A' is given by :

$$\text{M.D. (about an average A)} = \frac{\sum f|x-A|}{N}$$

where x is the mid-value of the class interval and $f|x-A|$ is the absolute value of the deviation of the given value of x from the average 'A' multiplied by the corresponding frequency 'f' and $N = \sum f$.

Example 9 : Calculate the MD from (i) Mean and (ii) Median for the following data and also find its coefficients.

Marks :	0-10	10-20	20-30	30-40	40-50
No. of Students :	4	6	8	5	2

Solution : (i) For computation of MD from mean we prepare the following table.

Marks C.I.	Mid-value x	f	$d' = \frac{x-A}{i}$	fd'	$ x-\bar{x} $	$f x-\bar{x} $
0-10	5	4	-2	-8	18	72
10-20	15	6	-1	-6	8	48
20-30	25	8	0	0	2	16
30-40	35	5	1	5	12	60
40-50	45	2	2	4	22	44
		$\sum f = N = 25$	$\sum fd' = -5$			$\sum f x-\bar{x} = 240$

Let the assumed mean be 25

$$\begin{aligned}\text{Mean } (\bar{x}) &= A + \frac{\sum fd'}{N} \times i \\ &= 25 + \frac{(-5)}{25} \times 10 \\ &= 25 - 2 \\ &= 23\end{aligned}$$

$$\begin{aligned}\therefore \text{M.D. (about mean)} &= \frac{1}{N} \sum f|x - \bar{x}| \\ &= \frac{240}{25} \\ &= 9.6 \text{ marks.}\end{aligned}$$

Coefficient of MD (about mean)

$$\begin{aligned}&= \frac{\text{MD}}{\text{Mean}} \\ &= \frac{9.6}{23} \\ &= 0.42\end{aligned}$$

(ii) For computation of MD from median, we prepare the following table.

Marks C.I.	Mid-value x	f	c.f.	x-Median = x-23.13	f x-23.13
0-10	5	4	4	18.13	72.52
10-20	15	6	10	8.13	48.78
20-30	25	8	18	1.87	14.96
30-40	35	5	23	11.87	59.35
40-50	45	2	25	21.87	43.74
		$\sum f = N = 25$			$\sum f x-23.13 $ = 239.35

Median = value of $\left(\frac{N}{2}\right)^{\text{th}}$ observation

= value of $\left(\frac{25}{2}\right)^{\text{th}}$ or 12.5th observation

The c.f. just greater than 12.5 is 18 which corresponds to the class interval 20–30.

∴ Median lies in the class 20–30

$$\text{Median} = L + \frac{\frac{N}{2} - \text{c.f.}}{f} \times i$$

Here, $L=20$, $\frac{N}{2}=12.5$, $\text{c.f.}=10$, $f=8$ & $i=10$

$$\begin{aligned} \therefore \text{Median} &= 20 + \frac{12.5 - 10}{8} \times 10 \\ &= 20 + 3.13 \\ &= 23.13 \end{aligned}$$

$$\begin{aligned} \therefore \text{M.D. (about median)} &= \frac{\sum f|x - \text{Median}|}{N} \\ &= \frac{239.35}{25} \\ &= 9.57 \text{ marks} \end{aligned}$$

Coefficient of MD (about median)

$$\begin{aligned} &= \frac{\text{MD}}{\text{Median}} \\ &= \frac{9.57}{23.13} \\ &= 0.41 \end{aligned}$$

5.7.2 MERITS, DEMERITS AND USES OF M.D.

Merits

1. It is rigidly defined.
2. It is easy to understand and calculate.
3. It provides a better measure for comparison.

Demerits.

1. It cannot be calculated for open-end class intervals.
2. It ignores the signs of the deviations.
3. It is not capable of further algebraic treatment.

Uses

M.D. and its coefficients are used in studying economic problems such as distribution of income and wealth.

EXERCISE (5 C)

- What do you mean by M.D. Write down its merits and demerits.
- Discuss the relative merits of M.D. over range and Q.D. as a measure of dispersion.
- Calculate the M.D. from mean and median for the following data.
 - 90, 110, 120, 115, 100 [Ans: 9.6, 9]
 - 100, 150, 200, 250, 360, 490, 500, 600, 671 [Ans: 174.44, 173.44]
- Calculate MD from (i) mean and (ii) median for the following data and also find its coefficient:

x :	4	6	8	10	12	14	16
f :	2	1	3	6	4	3	1

[Ans: 2.44, 2.4, 0.239, 0.24]
- Calculate MD about mean for the following data and also find its coefficient

Weight (lb) :	95–105	105–115	115–125	125–135
No. of Persons :	20	26	38	16

[Ans: 8.6 lbs, 0.075]
- Calculate MD from median for the following distribution of the scores of 50 students :

Scores :	140–150	150–160	160–170	170–180	180–190	190–200
No. of Students :	4	6	10	10	9	3

[Ans: 10.24 scores]
- Find the MD from Median.

Marks	No. of Students
less than 80	100
less than 70	90
less than 60	80
less than 50	60
less than 40	32
less than 30	20
less than 20	13
less than 10	5

[Ans: 14.28]

8. Calculate MD from median for the following distributions :

Wages (in Rs.) :	125	175	225	275	325
Mid-values :	3	8	21	6	2

[Ans: Rs.31.61]

5.8 STANDARD DEVIATION (S.D.) :

Standard deviation is defined as the positive square root of the arithmetic mean (AM) of the squares of the deviations of the given observations from their AM. It is denoted by the greek alphabet σ (sigma)

If x_1, x_2, \dots, x_n is a set of 'n' observations then its S.D. is given by :

$$\sigma = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$$

Where $\bar{x} = \frac{\sum x_i}{n}$ is the AM of the given values.

$$\begin{aligned} \text{Alternatively, } \sigma &= \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2} \\ &= \sqrt{\frac{1}{n} \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2)} \\ &= \sqrt{\frac{\sum x_i^2}{n} - \frac{2\sum x_i}{n} \bar{x} + \frac{1}{n} \sum \bar{x}^2} \\ &= \sqrt{\frac{\sum x_i^2}{n} - 2\bar{x} \cdot \bar{x} + \bar{x}^2} \\ &= \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2} \\ \therefore \sigma &= \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} \quad \left[\because \bar{x} = \frac{1}{n} \sum x_i \right] \end{aligned}$$

The square of S.D. is called variance

$$\therefore \text{Variance} = \sigma^2$$

Example 10 : Find the S.D. of 1, 2, 3, 4

Solution :
$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

Here, $n = 4$

$$\sum x = 1+2+3+4 = 10$$

$$\sum x^2 = 1+4+9+16 = 30$$

$$\therefore \sigma = \sqrt{\frac{30}{4} - \left(\frac{10}{4}\right)^2} = \sqrt{7.5 - 6.25}$$

$$= 1.12 \text{ Ans.}$$

A relative measure is coefficient of S.D. and is given by :

$$\text{Coefficient of S.D.} = \frac{\sigma}{x}$$

Note :

1. S.D. is considered to be the best measure of dispersion.
2. Coefficient of S.D. is the most used measure of dispersion for comparing the homogeneity and heterogeneity of two or more distributions.
3. S.D. is also known as Root Mean Square Deviation.

5.8.1 CALCULATION OF STANDARD DEVIATION

A. Calculation of S.D. for individual series :

There are two methods of calculating S.D. for the individual series, they are :

- (i) by taking deviations of the observations from actual mean.
- (ii) by taking deviations of the observations from an assumed mean.
- (iii) Deviations taken from actual mean :

This method is known as direct method.

Here the S.D. can be obtained by the formula

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} \\ &= \sqrt{\frac{d^2}{n}}, \text{ when } d = x_i - \bar{x}, \text{ and} \end{aligned}$$

x_i is the value of the observations,

\bar{x} = AM and n is the total number of observations.

(ii) Deviations taken from assumed mean :

When actual mean is in fraction, we use this method. The formula is

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

Where $d = x_i - A$, A = assumed mean, n = total number of observations.

Example 11 : From the following data calculate the S.D. and the variance.

12, 15, 16, 18, 19, 20

Solution : Let us take the assumed mean 'A' as 17. Now to calculate SD let us prepare the following table.

x	d = (x - A) = (x - 17)	d ²
12	-5	25
15	-2	4
16	-1	1
18	1	1
19	2	4
	$\sum d = -5$	$\sum d^2 = 35$

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \\ &= \sqrt{\frac{35}{5} - \left(\frac{-5}{5}\right)^2} = \sqrt{7 - 1} = \sqrt{6}\end{aligned}$$

$$\therefore \sigma = 2.449$$

$$\text{Variance} = \sigma^2 = (2.449)^2 = 5.99$$

B. Calculation of S.D. from ungrouped frequency distribution :

The S.D. for ungrouped frequency distribution can be calculated by the following methods.

- (i) Deviations taken from actual mean
- (ii) Deviations taken from assumed mean
- (iii) Step deviation method.

(i) Deviations taken from actual mean :

The formula is given by

$$\sigma = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}}$$

where \bar{x} is the mean; x is the value of variable, f is the corresponding frequency, $N = \sum f$.

(ii) Deviations taken from assumed mean :

The formula is given by :

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

where $d = x - A$, A is the assumed mean, $N = \sum f$.

Example 12 : Find the S.D. and its coefficient from the following data :

x :	1	2	3	4	5	6
f :	2	5	9	16	7	3

Solution : For calculation of S.D., we must prepare the following table.

x	f	d=(x-A) =(x-4)	fd	fd ²
1	2	-3	-6	18
2	5	-2	-10	20
3	9	-1	-9	9
4	16	0	0	0
5	7	1	7	7
6	3	2	6	12
	$\sum f = N = 42$		$\sum fd = -12$	$\sum fd^2 = 66$

Let the assumed mean be 4

$$\therefore \sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= \sqrt{\frac{66}{42} - \left(\frac{-12}{42}\right)^2}$$

$$= \sqrt{1.57 - 0.08}$$

$$= \sqrt{1.49} = 1.22$$

$$\therefore \sigma = 1.22 \text{ Ans.}$$

$$\text{Coefficient of SD} = \frac{\sigma}{\bar{x}}$$

$$\text{we have } \bar{x} = A + \frac{\sum fd}{N}$$

$$= 4 + \frac{(-12)}{42}$$

$$= 4 - 0.29$$

$$= 3.71$$

$$\therefore \text{Coefficient of SD} = \frac{\sigma}{\bar{x}} = \frac{1.22}{3.71}$$

$$= 0.329 \text{ Ans.}$$

(iii) Step Deviation Method : is used if there is a common difference between the values of the variable.

The formula is as given below

$$\sigma = \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \times i$$

Where, $d' = \frac{x-A}{i}$, 'A' is the assumed mean and 'i' is the common factor.

Example 13 : Calculate SD from the following data :

Wages (in Rs.) :	10	20	30	40	50	60
No. of Workers :	2	5	9	6	3	1

Solution : For computation of SD by step deviation method, we prepare the following table :

Wages (x)	No. of Workers (f)	$d' = \frac{x-A}{i}$ $= \frac{x-30}{10}$	fd'	fd'^2
10	2	-2	-4	8
20	5	-1	-5	5
30	9	0	0	0
40	6	1	6	6
50	3	2	6	12
60	1	3	3	9
	$\Sigma f = N = 26$		$\Sigma fd' = 6$	$\Sigma fd'^2 = 40$

Let the assumed mean A be 30 and common factor $i = 10$.

$$\begin{aligned}\sigma &= \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times i \\ &= \sqrt{\frac{40}{26} - \left(\frac{6}{26}\right)^2} \times 10 \\ &= \sqrt{1.54 - 0.05} \times 10 \\ &= \sqrt{1.49} \times 10 \\ &= 1.22 \times 10 = 12.20 \\ \sigma &= 12.20 \text{ Ans.}\end{aligned}$$

C. Calculation of S.D. for grouped or continuous frequency distribution :

The method is almost same as the step deviation method for simple frequency distribution, the only difference is that the mid values of class intervals are to be found out.

The formula is

$$\sigma = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times i$$

Where, $d' = \frac{x-A}{i}$, x is the mid values of the class intervals, A is the assumed mean and i is the width of the class intervals.

f = frequency.

$N = \sum f$ = total frequency.

Example 14 : Calculate the SD from the following data :

Daily wages :	5-15	15-25	25-35	35-45	45-55	55-65
(in Rs.)						
No. of Workers :	2	5	9	6	3	1

Solution : For computation of SD we prepare the following table :

Daily wages	Mid-value	No. of workers	$d' = \frac{x-A}{i}$ $= \frac{x-30}{10}$	fd'	fd'^2
X.I.	x	f			
5-15	10	2	-2	-4	8
15-25	20	5	-1	-5	5
25-35	30	9	0	0	0
35-45	40	6	1	6	6
45-55	50	3	2	6	12
55-65	60	1	3	3	9
		$\sum f = N = 26$		$2fd' = 6$	$\sum fd'^2 = 40$

Let the assumed mean A be 30.

$$\sigma = \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \times i$$

$$= \sqrt{\frac{40}{26} - \left(\frac{6}{26}\right)^2} \times 10$$

$$= \sqrt{1.49} \times 10$$

$$\sigma = 12.20 \text{ Ans.}$$

5.8.2 PROPERTIES OF STANDARD DEVIATION :**Property 1 :** Combined SD

The combined S.D. of two sets of data containing n_1 and n_2 observations with means \bar{x}_1 and \bar{x}_2 and standard deviations σ_1 and σ_2 respectively is given by.

$$\sigma_{12} = \sqrt{\frac{n_1(\sigma_1^2 - d_1^2) + n_2(\sigma_2^2 - d_2^2)}{n_1 + n_2}}$$

Where , σ_{12} = combined S.D. $d_1 = \bar{x}_1 - \bar{x}_{12}$, $d_2 = \bar{x}_2 - \bar{x}_{12}$ and $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

The formula for combined S.D. of two sets of data can be extended to compute the S.D. of more than two sets of data.

Property 2 : S.D. of natural numbers :

The S.D. of first 'n' natural numbers is given by:

$$\sigma = \sqrt{\frac{1}{12}(n^2 - 1)}$$

Property 3 : S.D. calculated from two values x_1 and x_2 of the variable x is equal to half their difference, ie.

$$\sigma = \frac{1}{12}(x_1 - x_2)$$

Property 4 : S.D. is independent of change of origin, but is dependent on the change of scale. That is, if each observation is increased or decreased by any number, will find the same S.D.

If each observation is multiplied or divided by a constant, S.D. will be similarly affected.

If $y = ax + b$, ($a > 0$)

We have $\bar{y} = a\bar{x} + b$

and $y - \bar{y} = ax + b - a\bar{x} - b = a(x - \bar{x})$

$$\begin{aligned} \text{Now } \sigma_y^2 &= \frac{1}{n} \sum (x - \bar{y})^2 \\ &= \frac{a^2 \sum (x - \bar{x})^2}{n} \end{aligned}$$

$$= a^2 \sigma_x^2$$

$$\therefore \sigma_y = a\sigma_x$$

$$\text{or, } \sigma_{ax+b} = a\sigma_x \quad [y = ax + b]$$

This, implies that S.D. is independent of origin, but dependent on scale.

Example 15 : The mean and SD of distribution of marks of 60 students of college A are 50 and 5 and that of 90 students of college B are 60 and 8 respectively. Find the S.D. of all 150 students taken together.

Solution : Here $n_1 = 60$, $\bar{x}_1 = 50$, $\sigma_1 = 5$
& $n_2 = 90$, $\bar{x}_2 = 60$, $\sigma_2 = 8$

$$\begin{aligned}\therefore \text{Combined mean } \bar{x}_{12} &= \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2} \\ &= \frac{60 \times 50 + 90 \times 60}{60 + 90} \\ &= \frac{3000 + 5400}{150} \\ &= 56\end{aligned}$$

$$\text{Combined S.D. : } \sigma_{12} = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

$$\text{Where, } d_1 = \bar{x}_1 - \bar{x}_{12} = 50 - 56 = -6$$

$$d_2 = \bar{x}_2 - \bar{x}_{12} = 60 - 56 = 4$$

$$\begin{aligned}\therefore \sigma_{12} &= \sqrt{\frac{60(5^2 + (-6)^2) + 90(8^2 + 4^2)}{60 + 90}} \\ &= \sqrt{\frac{60(25 + 36) + 90(64 + 16)}{150}} \\ &= 8.51 \text{ Ans.}\end{aligned}$$

Example 16 : If S.D. of x is 10, find the S.D. of (i) $x+5$, (ii) $x-4$, (iii) $(-6x)$, (iv) $\frac{x}{5}$, (v) $2x+3$.

Solution : Since S.D is independent of origin, so addition and subtraction of a constant does not change the S.D of x .

$$\therefore \text{(i) S.D. of } (x+5) = \text{S.D. of } x = 10$$

$$\text{(ii) S.D. of } (x-4) = \text{S.D. of } x = 10$$

As S.D. is dependent on scale of origin, so multiplication and division of S.D. by a constant affects the S.D. S.D. is always positive so negative sign is ignored.

(iii) S.D. of $(-6x) = 6$ (S.D of x) = $6 \times 10 = 60$ [negative sign is ignored]

(iv) S.D of $(\frac{x}{5}) = \frac{1}{5}$ (S.D of x) = $\frac{10}{5} = 2$

(v) S.D of $(2x+3) = 2$ (S.D of x) = $2 \times 10 = 20$

5.8.3 MERITS, DEMERITS AND USES OF S.D.

Merits

1. It is rigidly defined.
2. It is based on all the observations.
3. It is suitable for further mathematical treatment.
4. It is least affected by fluctuations of sampling.

Demerits

1. It is not easy to understand and calculate.
2. It gives more weight to extreme values.

Uses

1. S.D. is used widely in statistical theory such as in skewness, kurtosis, correlation and regression analysis, sampling theory has made it the most powerful measure of dispersion.
2. It is also used in biological studies.

Note 1: S.D. is considered to be the best measure of dispersion as it satisfies almost all the characteristics laid down for an ideal measure of dispersion.

Note 2: In a moderately skewed distribution : $4SD = 5MD = 6QD$

5.8.4 COEFFICIENT OF VARIATION (C.V.)

The coefficient of variation is defined as the percentage variation in mean, while standard deviation is considered as the total variation in the mean. The coefficient of dispersion based on standard deviation multiplied by 100 is called the coefficient of variation and abbreviated as C.V. Thus,

$$C.V. = \frac{\sigma}{x} \times 100$$

Coefficient of variation suggested by Prof. Karl Pearson is considered as the best relative measure of dispersion. C.V. is always expressed in percentage form.

For comparing the variability of two distributions we compute the coefficient of variation for each distribution. A distribution with smaller C.V. is said to be more homogeneous or less variable than the other, and the distribution with greater C.V. is said to be more heterogeneous or more variable than the other.

Example 17 : Two workers on the same job show the following results over a long period of time.

	Worker A	Worker B
Mean time of completing the job (in minutes)	30	25
S.D. (in minutes)	6	4

Which workers appears to be more consistant in the time he require to complete the job.

Solution : Here we use C.V. for comparision

$$\begin{aligned} \text{Now, C.V. of worker A} &= \frac{\sigma_1}{x_1} \times 100 \\ &= \frac{6}{30} \times 100 \\ &= 20\% \end{aligned}$$

$$\begin{aligned} \text{C.V. of worker B} &= \frac{\sigma_2}{x_2} \times 100 \\ &= \frac{4}{25} \times 100 \\ &= 16\% \end{aligned}$$

\therefore Worker B appears to be more consistant.

Example 18 : Calculate the coefficient of variation from the following data :

Marks :	0–10	10–20	20–30	30–40	40–50
No. of Students :	4	5	6	3	2

Solution : For calculation of C.V. we prepare the following table :

Marks	Mid value x	f	$d' = \frac{x-A}{i} = \frac{x-25}{10}$	fd'	fd ²
0–10	5	4	–2	–8	16
10–20	15	5	–1	–5	5
20–30	25	6	0	0	0
30–40	35	3	1	3	3
40–50	45	2	2	4	8
Total		$\Sigma f = N = 20$		$\Sigma fd' = -6$	$\Sigma fd^2 = 32$

Let the assumed mean A be 25.

$$\text{C.V.} = \frac{\sigma}{\bar{x}} \times 100$$

$$\begin{aligned} \text{Here } \bar{x} &= A + \frac{\sum fd'}{N} \times i \\ &= 25 + \frac{(-6)}{20} \times 10 \\ &= 22 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum fd'^2}{N} - \left(\frac{\sum fd'}{N}\right)^2} \times i \\ &= \sqrt{\frac{32}{20} - \left(\frac{-6}{20}\right)^2} \times 10 \\ &= \sqrt{1.6 - 0.09} \times 10 \\ &= 12.29 \end{aligned}$$

$$\therefore \text{C.V.} = \frac{12.29}{22} \times 100 = 55.86$$

$$\text{CV} = 55.86 \text{ Ans.}$$

Example 19 : The mean and standard deviation of a series of 20 items were calculated by a student as 20 cm and 5 cm respectively. But while calculating them an item 13 was misread as 30. Find the correct mean and S.D.

Solution : We have, $n = 20$, $\bar{x} = 20$ cm & $\sigma = 5$ cm

$$\bar{x} = \frac{\sum x}{n}$$

$$\Rightarrow \sum x = n \bar{x} = 20 \times 20 = 400 \text{ cm}$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$\text{or, } 5 = \sqrt{\frac{\sum x^2}{20} - \left(\frac{400}{20}\right)^2}$$

$$\text{or, } 25 = \frac{\sum x^2}{20} - 400$$

$$\text{or, } \frac{\sum x^2}{20} = 25+400 = 425$$

$$\text{or } \sum x^2 = 425 \times 20 = 8500$$

$$\text{Corrected } \sum x = 400 - 30 + 13 = 383$$

$$\begin{aligned} \text{Corrected } \sum x^2 &= 8500 - (30)^2 + (13)^2 \\ &= 8500 - 900 + 169 = 8269 \end{aligned}$$

$$\therefore \text{ Corrected mean } (\bar{x}) = \frac{383}{20} = 19.15 \text{ Ans.}$$

$$\begin{aligned} \text{Corrected SD } (\sigma) &= \sqrt{\frac{8269}{20} - (19.15)^2} \\ &= \sqrt{413.45 - 366.72} = \sqrt{46.73} \\ &= 6.83 \text{ Ans.} \end{aligned}$$

EXERCISE (5 D)

- Define S.D. Why SD is called the ideal measure of dispersion.
- Write down the merits and demerits of S.D.
- What is coefficient of variation? What purpose does it serve?
- In what situation would you prefer C.V. as a measure of dispersion.
- Find the S.D. of the following :
 - 1, 2, 3, 4, 5 [Ans: 1.414]
 - 16, 13, 17, 22 [Ans: 3.2]
- If S.D. of 1, 2, 3,.....,n be $\sqrt{14}$, find n. [Ans: 13]
- Find the S.D. and its coefficient from the following data :

x :	10	11	12	13	14	15	16
f :	2	7	11	15	10	4	1

[Ans: 1.342, 0.104]
- Calculate SD for distribution of marks given below :

Marks :	10	20	30	40	50	60	70
No. of Students :	10	8	12	20	10	6	4

[Ans: 16.7 marks]

9. Calculate the SD and its coefficient from the following data :

C.I. :	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f:	8	12	17	14	9	7	4

[Ans: 16.67, 0.54]

10. In a moderately skewed distribution, the MD is 60, find the value of SD and QD.

[Ans: 75, 50]

11. The mean and SD of distribution of 100. Observations are 50 and 5, and that of 150 observations are 40 and 6 respectively. Find the SD of all 250 observations taken together.

[Ans: 7.46]

12. Find the coefficient of variation of the number 1, 2, 3, 4, 5.

[Ans: 47.13%]

13. Following data is given for two factories. Find out in which factory there is more variability in wages.

	Factory I	Factory II
No. of workers :	50	60
Mean monthly wage :	Rs. 60	Rs. 48
Variance :	100	144

[Ans: Factory II]

14. Calculate mean, SD and variance from the following data :

Class Interval :	60-62	63-65	66-68	69-71	72-74
Frequency :	5	18	42	27	8

[Ans: 67.45, 2.92, 8.53]

15. Find the mean, SD and CV of the following data :

Marks :	5-10	10-15	15-20	20-25	25-30	30-35
No. of Students :	2	9	29	54	11	5

[Ans: 21.05, 4.84, 23%]

16. Calculate variance and CV for the distribution of marks given below :

Marks	No. of Students.
less than 10	3
less than 20	7
less than 30	13
less than 40	18
less than 50	20

[Ans: 144.72, 48.22%]

17. The mean and SD of 100 observations are found to be 30 and 10 respectively. Two observations were taken by mistake as 12 and 31 instead of 21 and 13 respectively. Find the correct mean and SD.

[Ans: 29.91, 10.02]

18. The mean and standard deviation of a group of 25 observations were found to be 30 and 3 respectively. After the calculation it was found that two of the observations were incorrect, which were recorded as 29 and 31. Find the mean and SD, if the incorrect observations are excluded.

[Ans: 30, 3.11]

19. The mean of five observations is 4.4 and the variance is 8.24. If three of the five observations are 1, 2 and 6, find the other two.

[Ans: 9, 4]

20. Find the mean and variance of the following data :

Measurement	No. of articles
More than 80	5
More than 70	14
More than 60	34
More than 50	65
More than 40	110
More than 30	150
More than 20	170
More than 10	176
More than 0	180

[Ans: 45.2, 282.95]

5.8.5 LORENZ CURVE

Dr. Max O Lorenz, an economic statistician used a graphic method for measuring dispersion. Hence it is known as Lorenz Curve. It was used for studying the economic inequalities such as inequalities in the distribution of income and wealth between different countries or between different periods of time. At present, Lorenz Curve is also used in business to study the differences of the distribution of wages, profits, production, population etc.

Lorenz Curve is also known as Cumulative Percentage Curve. It deals with cumulative percentage values of the variable and the cumulated percentage of frequencies. Here on the X-axis we take the cumulated percentage of variable and on the Y-axis we take the cumulated percentage of frequency.

First, the values of the variables and the frequencies are both cumulated. Then the percentage are calculated for the cumulated values of the variables and the corresponding frequencies. These percentages are to be plotted in the graph paper. If wealth are equally distributed among the people concerned, the curve would be a straight line joining the extreme points of the different scales ie. (0, 0) and (100, 100). This line is known as Line of equal distribution.

If the distribution of items are not proportionately equal, it indicates variability and the curve would be away from the line of equal distribution. The greater the variability the greater is the distance of the curve from the line of equal distribution. So, if the curve is closer to the line of equal distribution it indicates smaller variability

Lorenz Curve given us only a relative idea of the dispersion as compared to the line of equal distribution. It does not provide any numerical value of the variability for the given distribution.

Example 20 : Draw Lorenz Curve for the following distribution and compare the variability of the two distributions.

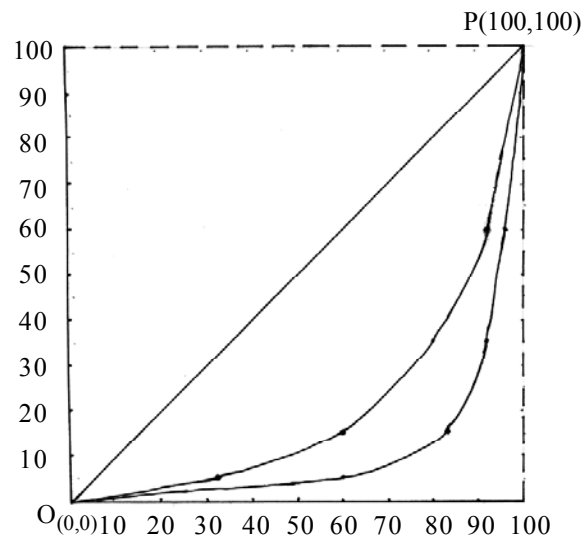
Profit earned (in Rs.1000)	No. of Companies	
	Area A	Area B
10	8	15
20	7	6
40	5	2
50	3	1
80	2	1

Solution : For drawing the Lorenz Curve we must prepare the following table :

Profit earned (in Rs.1000)			Area A			Area B		
Profit (1000 Rs.)	Cumulated Profit Profit	Cumulated Percentage	No. of Companies Companies	Cumulated No. of	Cumulated Percentage No. of Companies	No. of Companies	Cumulated No. of Companies	Cumulated Percentage No. of Companies
10	10	5	8	8	32	15	15	60
20	30	15	7	15	60	6	21	84
40	70	35	5	20	80	2	23	92
50	120	60	3	23	92	1	24	96
80	200	100	2	25	100	1	25	100

Now plotting the percentage of cumulated profit earned in the Y-axis and the percentage of cumulated number of companies of area A and area B in the X-axis we get the Lorenz Curve as follows :

The curve B is further away from the line of equal distribution than A. Therefore there is greater variability in the profit earned by the companies of area B than area A.



EXERCISE (5 E)

1. What is Loreng Curve? What is its use?
2. Write a short note on Lorenz Curve.
3. Draw Lorenz Curve for the following distribution and compare the variability of the two distributions.

Profit earned (in Rs.)	No. of Companies	
	Area A	Area B
6	6	2
25	11	38
60	13	52
84	14	28
105	15	38
150	17	26
170	10	12
400	14	4

4. State whether the statements are true or false.

- (i) Range is a relative measure of dispersion.
- (ii) Q.D. is also called semi interquartile range.
- (iii) MD is minimum when calculated from mode.
- (iv) Mean, SD and CV have the same unit.
- (v) Variance is always non-negative.
- (vi) S.D. is the best measure of dispersion.

[Ans: (i) False, (ii) True, (iii) False,
(iv) False, (v) True, (vi) True]

5. Fill in the blanks :

- (i) The concept of Lorenz Curve was put forward by _____.
- (ii) C.V. is _____ of unit of measurement.
- (iii) Variance is equal to the square of _____.
- (iv) S.D. is always _____ than range.
- (v) S.D. is _____ of the change of origin.
- (vi) The further the Lorenz Curve is from the line of equal distribution the _____ is the variability in the series.

[Ans: (i) Max O. Lorenz, (ii) free, (iii) S.D.,
(iv) less, (v) independent, (vi) greater]

UNIT-6**CORRELATION**

In our previous studies of data analysis with the help of measures of central tendency and dispersion etc. We came across data involving only one variable. These type of data are known as univariate data. For example data relating to marks, age, weight, income etc. are univariate data. But in practice we may face data involving two or more variables. Data involving two variables are known as bivariate and data involving more than two variables are known as multivariate data. A few examples of bivariate data are : (i) height and weight of a group of persons (ii) income and expenditure of a group of families. (iii) price and demand of commodity for the last 10 years etc. If a bivariate data is given for analysis, our interest will go at first to know if there is any relationship between the two variables. The knowledge of such a relationship is important to draw inferences about the bivariate distribution correlation thus is a statistical tool which studies the relationship between the two variables and it involves various methods for studying and measuring the extent and degree of relationship.

DEFINITION :

If it is observed in a bivariate data that the change in the values of our variable results in a corresponding change in the values of the other variable, then the two variables under study are said to have correlation.

TYPES OF CORRELATION :

Depending on the nature of change correlation has been classified as follows :

1. Positive correlation
2. Negative correlation
3. Perfect correlation
 - (i) Perfect positive correlation
 - (ii) Perfect negative correlation

1. Positive correlation :

If it is seen in a bivariable data that the increase (decrease) in the values of our variable results in a corresponding increase (decrease) in the values of the other variable, then the correlations between the two variable under study is said to be positive. For example, the correlation between (i) income and expenditure (ii) height and weight of a group of persons is positive.

Thus in case of positive correlation changes in the values always occur in the same direction.

Since the changes in both x and y are occurring in the same direction and the corresponding changes are proportional. So the correlation between the two variables x and y is perfect positive.

Again if the changes in both the variables occur in the opposite direction and the corresponding changes bear a constant ratio, then the correlation is said to be perfect negative.

For example, consider the following bivariate distributions :

<u>Decrease</u> →		← <u>Increase</u>					
x : 50	48	47	43	40	38	34	29

<u>Increase</u> →		← <u>Decrease</u>					
y : 4	10	13	25	34	40	52	67

$$\frac{\text{Amount of change in } x}{\text{Amount of change in } y} : \frac{2}{6} = \frac{1}{3} = \frac{4}{12} = \frac{3}{9} = \frac{2}{6} = \frac{4}{12} = \frac{5}{15} = \frac{1}{3}$$

Since the changes in x and y are occurring in the opposite direction and the corresponding changes bear a constant ratio, therefore, the correlation between x and y is perfect negative.

Remarks :

1. If the data given in the examples of perfect positive and perfect negative correlation are plotted on a graph paper, we will find that all the points in both the cases will lie on a straight line with different directions.
2. Any deviation of the points from the straight lines measures the extent or degree of linear relationship between the two variables under study.
3. Our present discussion is restricted to only linear relationship between two variables.

METHODS OF STUDYING CORRELATION :

Among the various methods which ascertain only linear relationship between two variables, we consider only the following two methods :

- I. Scatter diagram and
- II. Karl Pearson's coefficient of correlation.

I. Scatter Diagram :

This is the simplest method of studying whether there is any relationship between the two variables in a given bivariate distribution. In this method each pair value of the variables of the given data is plotted on a graph paper using dot marks and thus obtain as many dots as the number of pairs of observations. The diagram of dots so obtained is

known as scatter diagram. By looking at the scatterness of the data we can have a fair idea as to whether the variables are related or not. The greater is the scatter of the dots, the lesser is the relationship between the two variables and vice-versa.

In the following we discuss considering different types of scatter diagram has correlation can be interpreted:

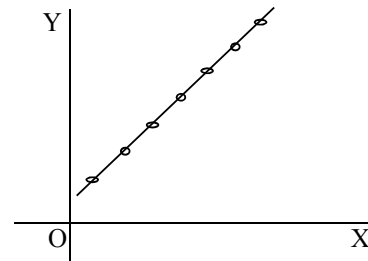


Diagram 6.1

We observe that the dots of the diagram 6.1 are such that a straight line can be made to pass through all the dots i.e.; all the plotted points lie on a straight line and the line is rising from lower left-hand corner to the upper right-hand corner. The proportional changes in the same direction of the values of two variables x and y in a bivariate data always exhibits this type of diagram and there by conclude that there is perfect positive correlation between the two variables x and y .

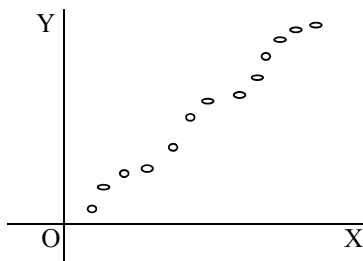


Diagram 6.2

The scatter diagram 6.2 reveals that the plotted points have an upward trend from lower left hand corner to the upper right hand corner. This type of tendency is seen when the changes in the values of the two variables occur in the same direction and the changes are not proportional then we conclude that there is a positive correlation between the two variables x and y under study.

In the above scatter diagram 6.3, we see that the dots are such that a straight line can be made to pass through all the dots i.e.; all the plotted points lie on a straight line which has a downward trend from upper left-hand corner to lower right-hand corner. The proportional changes in the opposite direction of the values of the two variables in a bivariate data always results in this type of diagram. The correlation in this case between the two variables under study is perfect negative.

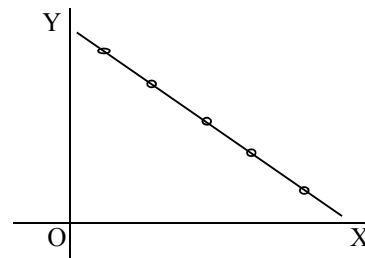


Diagram 6.3

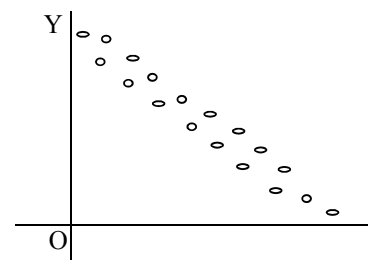


Diagram 6.4

The dots of diagram 6.4 shows a downward tendency from upper left-hand corner to the lower right-hand corner. So we conclude that the correlation between the two variables under study is negative since the values of the variables in this case changes in the opposite direction and the changes are not proportional.

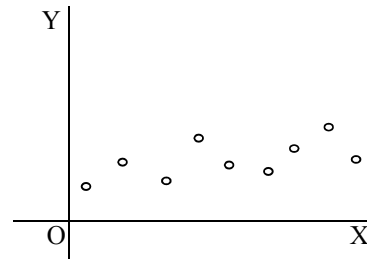


Diagram 6.5

In diagram 6.5, the dots reveal neither upward nor downward tendency rather the tendency is parallel to the x -axis. So we conclude that there is no correlation between the two variables under study.

Note :

As the scatter of the dots increases or closeness decreases from the lines given in diagrams 6.1 and 6.3 the degree of positive or negative correlation decreases and vice-versa.

MERITS AND DEMERITS OF SCATTER DIAGRAM METHOD :

Merits :

1. This method is the simplest method of studying correlation as it does not involve any mathematical complexities. It can be easily understood and it enables us to form a rough idea as to whether or not the variables under study are correlated.

Demerits :

1. This method can not measure the degree (high or low) of relationship between the two variables under study.
2. This method is not suitable if the number of observations is fairly large.

Example 1.1. : The marks of 10 students in Mathematics and Statistics are given as follows :

Marks in Maths x : 60 72 68 56 63 71 65 62 60 72

Marks in Stats. y : 58 66 64 51 55 63 62 57 64 68

Draw a scatter diagram of the above data and give your comment on the nature of correlation.

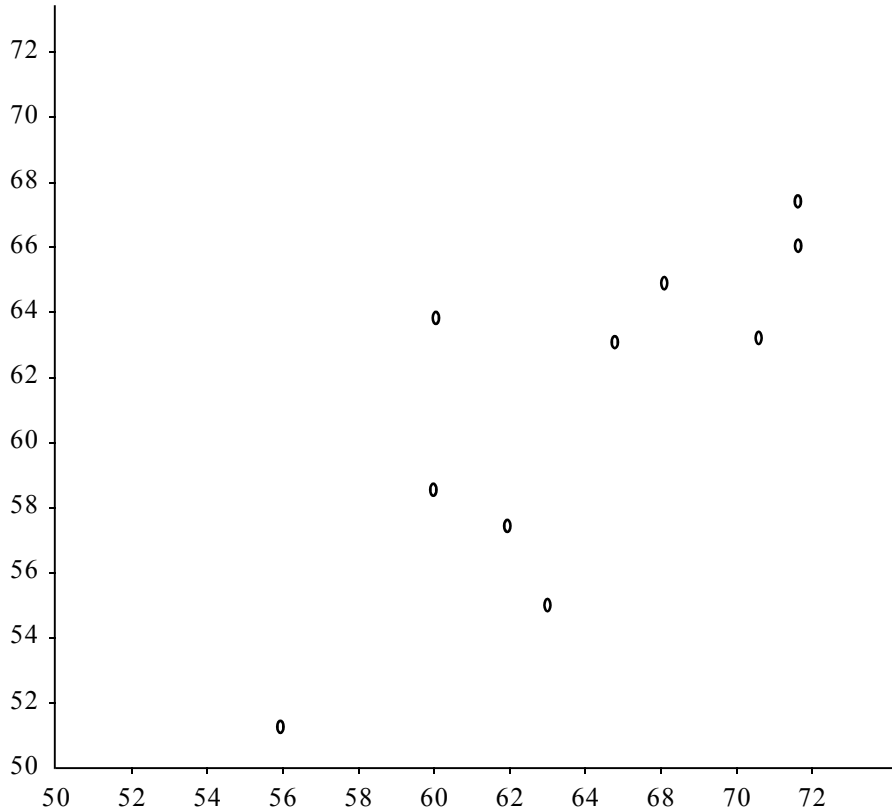


Diagram 6.6

From the above scatter diagram we observe that the plotted points shown an upward trend from the bottom of the left-hand corner to the top of the right hand corner. Hence we conclude from the closeness of the points that there is a fairly high degree of correlation between the marks. In mathematics and statistics of 10 students.

II. Karl Pearson’s Coefficient of Correlation :

Karl Pearson’s co-efficient of correlation is a mathematical method and it is most widely used method of numerical measurement of linear relationship between two variables. It is also known as product moment correlation.

If in a bivariate distribution $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are ‘n’ pairs of observations given of the variables x and y then Karl Pearson’s coefficient of correlation between x and y , denoted $r(x, y)$ or r_{xy} or r , is defined by

$$r = \frac{\text{cov}(x,y)}{r_x r_y} \dots\dots\dots (2.1)$$

Where $\text{cov}(x,y)$ known as covariance between x and y , is given by

$$\text{cov}(x,y) = \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) \dots\dots\dots (2.2)$$

σ_x and σ_y are known as standard deviations of x and y and are respectively given by

$$\sigma_x = \sqrt{\frac{1}{n} \sum (x - \bar{x})^2}$$

and
$$\sigma_y = \sqrt{\frac{1}{n} \sum (y - \bar{y})^2} \dots\dots\dots (2.3)$$

Note :

Σ represents summation being taken over ' n ' pairs of observations.

Substituting (1.2) and (1.3) in (1.1), we get

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \cdot \sum (y - \bar{y})^2}} \dots\dots\dots (2.4)$$

Properties of r :

- (i) ' r ' is a pure number i.e. it is free from any unit of measurement.
- (ii) The value of ' r ' as obtained by formula (1.4) always lies between -1 and $+1$ i.e., $-1 \leq r \leq +1$.
- (iii) Correlation coefficient between x and y is same as that of correlation coefficient between y and x i.e. $r(x, y) = r(y, x)$
- (iv) Correlation coefficient ' r ' is independent of change of origins and scale.

Interpretations of different values of ' r ' :

- (i) If $r = 0$ then we conclude that there is no linear relationship between the two variables under study.
- (ii) If $r = +1$, then we conclude that there is a perfect positive correlation between the two variables under study.
- (iii) If $r = -1$, then we conclude that there is a perfect negative correlation between the two variables under study.
- (iv) If $0 < r < 1$ then we conclude that there is positive correlation between the two

variable under study.

- (v) If $-1 < r < 0$, then we conclude that there is negative correlation between the two variables under study.
- (vi) If $r = 0.24$ and $r = 0.87$ then in both the cases there are positive correlation between the two variables and we conclude that in the former case there is low degree and in the later case there is high degree of positive correlation between the variables under study.
- (vii) If $r = -0.16$ and $r = -0.92$, then in both the cases there is negative correlation and to interpret have correctly we conclude that in the later case there is a high degree of negative correlation and in the former case there is a low degree of negative correlation between the two variables under study.

Remarks :

1. Karl Pearson’s coefficient of correlation only measures the linear relationship and their degree between two variables under study.
2. $r = 0$ indicates absence of linear relationship only but there may exist other types of non-linear relationship between the two variables under study.

Computation of Karl Pearson’s coefficient of correlation r :

The value of ‘ r ’ can be calculated by directly using the formula given in (2.4). In this case \bar{x} and \bar{y} to be calculated first and if the values of \bar{x} and \bar{y} comes out to be in decimals, then the entire calculation of ‘ r ’ will become labourious and will consume more time and will have a chance of committing mistake in the calculation. So to overcome this difficulty readers are suggested to use the simplified form of (2.4) for the calculation of r and it is given by

$$r = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sqrt{\left\{ \sum x^2 - \frac{(\sum x)^2}{n} \right\} \left\{ \sum y^2 - \frac{(\sum y)^2}{n} \right\}}} \dots\dots\dots (2.5)$$

Example 1 : Calculate Karl Pearson’s coefficient of correlation between the price and supply of a commodity during a period of 8 years from the following data.

Price (Rs. per kg) :	12	14	20	18	15	22	20	19
Supply (.00 kg) :	32	37	47	46	42	50	49	48

Solution by direct method using formula (1.4)

Let the price of the commodity be denoted by the variable X and the supply be denoted by Y.

CALCULATION OF CORRELATION COEFFICIENT

X	Y	$X - \bar{x}$ = $x - 17.5$	$Y - \bar{y}$ = $y - 43.9$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$	$(y - \bar{y})^2$
12	32	- 5.5	- 11.9	65.45	30.25	141.61
14	37	- 3.5	- 6.9	24.15	12.25	47.61
20	47	2.5	3.1	7.75	6.25	9.61
18	46	0.5	2.1	1.05	0.25	4.41
15	42	- 2.5	- 1.9	4.75	6.25	3.61
22	50	4.5	6.1	27.45	20.25	37.21
20	49	2.5	5.1	12.75	6.25	26.01
19	48	1.5	4.1	6.15	2.25	16.81
$\Sigma x = 140$	$\Sigma y = 351$			$\Sigma(x - \bar{x})(y - \bar{y})$ = 149.5	$\Sigma(x - \bar{x})^2$ = 84.0	$\Sigma(y - \bar{y})^2$ = 286.88

$$\begin{aligned} \bar{x} &= \frac{\sum x}{n} & \bar{y} &= \frac{\sum y}{n} \\ &= \frac{140}{8} & &= \frac{351}{8} \\ &= 17.5 \text{ (approx)} & &= 43.9 \text{ (approx)} \end{aligned}$$

$$\begin{aligned} \therefore r &= \frac{\sum(x - \bar{x})(y - \bar{y})}{\sqrt{\sum(x - \bar{x})^2 \cdot \sum(y - \bar{y})^2}} = \frac{149.5}{\sqrt{84.0 \times 286.88}} \\ &= \frac{149.5}{155.23} = 0.96 \text{ (approx)} \end{aligned}$$

Solution of Example 1 by direct method using simplified form (2.5) :

CALCULATION OF CORRELATION COEFFICIENT

X	Y	xy	x ²	y ²
12	32	384	144	1024
14	37	518	196	1369
20	47	940	400	2209
18	46	828	324	2116
15	42	630	225	1764
22	50	1100	484	2500
20	49	980	400	2401
19	48	912	361	2304
$\Sigma x=140$	$\Sigma y=351$	$\Sigma xy=6292$	$\Sigma x^2=2534$	$\Sigma y^2=15687$

$$\begin{aligned}
 r &= \frac{\Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\left\{ \Sigma x^2 - \frac{(\Sigma x)^2}{n} \right\} \left\{ \Sigma y^2 - \frac{(\Sigma y)^2}{n} \right\}}} \\
 &= \frac{6292 - \frac{140 \times 357}{8}}{\sqrt{\left\{ 2534 - \frac{(140)^2}{8} \right\} \left\{ 15687 - \frac{(351)^2}{8} \right\}}} \\
 &= \frac{149.5}{155.23} = 0.96
 \end{aligned}$$

The drawback of the simplified form (1.5) is that in case of values of X and Y given are big, the entire calculations become labourious and time consuming. We can overcome this difficulty in the following method.

Step-deviation method :

The aim of this method is to convert big values into small values and then correlation is calculated based on these small values. This method saves time and labour of calculation.

In this method the given values of X and Y are currented into u and v by the technique known as change of origin and scale as follows :

$$u = \frac{x-a}{h} \text{ and } v = \frac{y-b}{k}$$

When a = assumed mean of X

b = assumed mean of Y

h and k are the common factors and $h > 0$ and $k > 0$.

Then the correlation coefficient between X and Y is given by

$$r_{xy} = r_{uv} = \frac{\sum uv - \frac{(\sum u)(\sum v)}{n}}{\sqrt{\left\{ \sum u^2 - \frac{(\sum u)^2}{n} \right\} \left\{ \sum v^2 - \frac{(\sum v)^2}{n} \right\}}} \dots\dots\dots (2.6)$$

Which implies that correlation coefficient between X and Y is same as that of correlation coefficient between u and v .

Note : The relation (2.6) establishes the property (iv) of Karl Pearson's coefficient of correlation ' r '.

Solution of Example 1 by step-deviation method using formula (2.6) :

CALCULATION OF CORRELATION COEFFICIENT

$a = 18, b = 42, h = 1$ and $k = 1$

X	Y	$u = \frac{x-18}{1}$	$v = \frac{y-42}{1}$	uv	u^2	v^2
12	32	- 6	- 10	60	36	100
14	37	- 4	- 5	20	16	25
20	47	2	5	10	4	25
18	46	0	4	0	0	16
15	42	- 3	0	0	9	0
22	50	4	8	32	16	64
20	49	2	7	14	4	49
19	48	1	6	6	1	36
		$\sum u = - 4$	$\sum v = 15$	$\sum uv = 142$	$\sum u^2 = 86$	$\sum v^2 = 315$

$$\begin{aligned}
 r_{xy} = r_{uv} &= \frac{\sum uv - \frac{(\sum u)(\sum v)}{n}}{\sqrt{\left\{ \sum u^2 - \frac{(\sum u)^2}{n} \right\} \left\{ \sum v^2 - \frac{(\sum v)^2}{n} \right\}}} \\
 &= \frac{142 - \frac{(-4)(15)}{8}}{\sqrt{\left\{ 86 - \frac{(-4)^2}{8} \right\} \left\{ 315 - \frac{(15)^2}{8} \right\}}} \\
 &= \frac{149.5}{155.23} \\
 &= 0.96 \text{ (approx)}
 \end{aligned}$$

Note :

While selecting the assumed mean and common factor, the following points to be remembered:

1. For selecting the assumed mean values, first find the middle value of the concerned variable. If the middle value is not available amongst the values given, then choose a value from the given values which is nearest to the middle value and take that value as assumed mean.
2. For selecting the common factors, first look at the given values of the concerned variable if they are divisible by a common number or not. If divisible by a common number, then take that number as the common factor. Otherwise take common factor equal to 1 (one).

Example 2.1 : Calculate Karl Pearson's coefficient of correlation from the data given below :

X :	15	20	25	30	35	40	45	50	55
Y :	75	63	60	54	48	42	39	30	9

**CALCULATION OF CORRELATION COEFFICIENT
BY STEP-DEVIATION METHOD**

$a = 35, b = 42, h = 5$ and $k = 3$

X	Y	$u = \frac{x-35}{5}$	$v = \frac{y-42}{3}$	uv	u ²	v ²
15	75	-4	11	-44	16	121
20	63	-3	7	-21	9	49
25	60	-2	6	-12	4	36
30	54	-1	4	-4	1	16
35	48	0	2	0	0	4
40	42	1	0	0	1	0
45	39	2	-1	-2	4	1
50	30	3	-4	-12	9	16
55	9	4	-11	-44	16	121
		$\Sigma u = 0$	$\Sigma v = 14$	$\Sigma uv = -139$	$\Sigma u^2 = 60$	$\Sigma v^2 = 364$

$$\begin{aligned}
 r_{xy} = r_{uv} &= \frac{\Sigma uv - \frac{(\Sigma u)(\Sigma v)}{n}}{\sqrt{\left\{ \Sigma u^2 - \frac{(\Sigma u)^2}{n} \right\} \left\{ \Sigma v^2 - \frac{(\Sigma v)^2}{n} \right\}}} \\
 &= \frac{-139 - \frac{0 \times 14}{9}}{\sqrt{\left\{ 60 - \frac{(0)^2}{9} \right\} \left\{ 364 - \frac{(14)^2}{9} \right\}}} = \frac{-139}{\sqrt{60 \times 342.2}} \\
 &= \frac{-139}{143.29} = -0.97 \text{ (approx)}
 \end{aligned}$$

EXERCISE

1. **State whether the following statements are 'True' or 'False' :**

- (i) There are no limits to the value of ' r '.
- (ii) Pearson's coefficient of correlation is independent of change of origin but not of scale.
- (iii) If ' r ' is negative both the variables are decreasing.
- (iv) The unmarried value of r between two variables X and Y can not exceed unity.
- (v) If the value of ' r ' is positive i.e. $r > 0$, then as X increases, Y also increases.
- (vi) Karl Pearson's coefficient of correlation measures every type of relationship between the two variables.
- (vii) $r = 0$ means there is no relationship between the two variables.

[Ans : (i) F (ii) T (iii) F (iv) T (v) T (vi) F (vii) T (only lines)]

2. **Fill in the blanks :**

- (i) Positive correlation means on an average as one variable is increasing the variable is — and as one variable is decreasing the other is also—.
- (ii) The coefficient of correlation is independent of change of — and —.
- (iii) Karl Pearson's coefficient of correlation is a — number.
- (iv) $r = 0$ implies absence of — relationship between the two variables.
- (v) Karl Pearson's coefficient of correlation lies between — and —.
- (vi) If $r(x, y) = 1$, the line on the graph paper will move — from — to —.
- (vii) The absolute value of ' r ' is less than —.

[Ans. : (i) increasing, decreasing (ii) origin and scale (iii) pure (iv) linear (v) 0, 1 (vi) upward, left, right (vii) 1]

3. What is meant by correlation? Distinguish between positive, negative and zero correlation? Explain what is meant by perfect correlation.

4. Distinguish giving suitable examples between :

- (i) Positive and negative correlation.
- (ii) Linear and non-linear correlation.

5. (a) What is the meaning of coefficient of correlation?

- (b) What are the limits of the value r ? What do positive, negative and zero value of ' r ' indicate?

6. What is correlation? State the properties of Karl Pearson's coefficient of correlation.
7. What is a scatter diagram? How do you interpret a scatter diagram?
8. (a) What is a scatter diagram?
(b) How does it help in studying the correlation between two variables?
9. (a) What is correlation? What are the different types of correlation?
(b) Draw the scatter diagram of the different types of correlations.
10. (a) What is correlation? Distinguish between positive and negative correlation.
(b) Explain the significance of coefficient of correlation.
11. Define perfect correlation. Give numerical examples in each of the following cases.
 - (i) Positive correlation.
 - (ii) Negative correlation.
 - (iii) Perfect positive correlation, and
 - (iv) Perfect negative correlation.

12. The price and supplies of a commodity during a period of last 8 years are given in the following data:

Price (Rs. kg) :	10	12	18	16	16	20	18	20
Supply (100 kg) :	30	35	45	45	40	50	55	50

Calculate the coefficient of correlation between price and supply.

13. A group of 8 students got the following marks in statistics and accountancy :

Marks in stats :	50	60	65	70	75	40	70	80
Marks in Accy :	80	70	60	75	90	80	70	50

Calculate Karl Pearson's coefficient of correlation between the two sets of marks.

14. Draw the scatter diagram of the following data :

X :	15	20	18	12	40	60	20	80	70	75
Y :	40	30	50	30	20	10	30	60	64	70

Give your comment regarding the correlation between X and Y and the degree.

THEORY OF PROBABILITY

INTRODUCTION :

If we observe at the happenings of some phenomena related to business, economics, social science a even our day-to-day life, then we find that most of the occurrences of these phenomena are associated with the factor of uncertainty. For instance, a healthy man will survive for another six months cannot be predicted with certainty. Nothing can be said about the future life of an electric bulb after it has lasted for one year, a sales manager can never say with certainty that he will achieve his sales target in a financial year, an investment in share market will fetch an expected dividend can not be predicted with certainty. It is uncertain to predict a head on a tail in a toss of a perfect coin, the sex of a baby to be born can not be predicted with certainty. An element of uncertainty is associated with the savings that it may rain today. India may win the cricket match against Pakistan etc. Thus, we observed that the results in all the above phenomena can not be predicted with certainty and they are known probabilistic phenomena.

Now, the obvious question that arises into the mind of the readers is that – Are there phenomena whose occurrences can be predicted with certainty. The answer is yes. For instance, when two molecules of hydrogen and one molecule of oxygen are mixed up of water is formed in case of a perfect gas pressure X volume = constant, provided temperature remains constant or when potassium is added to water hydrogen gas is formed etc. are predictable phenomena as they give unique result under homogeneous conditions and they are known as deterministic phenomena.

Thus, most of the phenomena in physical and chemical sciences are deterministic and most of the phenomena in business economics and social sciences are probabilistic in nature.

Theory of probability is a very important branch of statistics which helps to measure numerically the uncertainty of occurrence of a phenomenon under study.

HISTORY :

Theory of probability had been originated while solving some problems of gambling related to games of chances for instance throwing of a dice, drawing cards from a pack of cards and so on. Galileo (1564-1642) an Italian mathematician was the first to attempt a quantitative approach to measure the probability or chance while dealing with some problems related to throwing of dice in gambling. The first foundation of the mathematical probability theory was laid in the mid seventeenth century by two French Mathematicians B. Pascal (1623-1662) and P. Fermat (1601-1665). While solving a number of problems posed by French Gambler and noble man Chevalier-De-Mere to Pascal considerable work in this field was done by Bernoulli (1654-1705) in his 'Treatise on Probabil-

ity' which was published in 1713. Others who made a significant contributions in developing the theory of probability are De-Moivre (1667-1754) Thomas Bayes (1702-1761). P.S. Laplace, (1749-1827) Most notable contributors to the modern theory of probability are Chebycher (1821-1894) A. Markoff (1856-1922), Liapomnoff A. Khinchive and A. N. Kolmogoror.

Theory of probability has been developed to such an extent that it is now used in all disciplines of social and natural sciences. It is used in the quantitative analysis of the problems related to business and economics. In insurance industry probability is extensively used to measure the risk or loss involved in order to calculate premium.

Approaches to measure Probability :

There are three different approaches to measure probability :

- (i) Classical a mathematical approach
- (ii) Empirical approach
- (iii) Axiomatri approach

Discussion of the measure of probability based on Empirical and Axiomative approaches one beyond the scope of the book.

Basic Terminology :

Different terms neded to define probability on discussed in this section.

Random Experiment :

It an experiment is conducted repeatedly essentially under homogeneous conditions and all possible results (out comes) of the experiment an known in advance but no result can be predicted with certainty then this type of experiment is known as random experiment.

For instance, (1) Tossing of a coin with two known possible outcomes viz; head a tail is a random experiment.

(2) Throwing of a die with six known possible outcomes viz; 1, 2, 3, 4, 5, and 6 (3) Drawing a card from a park of cards with 52 known possible outcome etc. an vandom experiments.

Trial and Event :

Performing a random experiment is a trial and getting an outcome or a combination of outcomes of the experiment is an event. Events an usually denoted by the capital alphabets A, B, C etc.

For example (1) tossing of a coin is a trial and getting a head in an event

(2) throwing of a die is a trial and getting 3 or getting an even number is an event.

Simple a Elementary and Compound a Composite events :

An event which can not be decomposed into two or more events in known as simple or elementary event. On the other hand if an event can be decomposed into more than one simple events then the event is known as compound a composite events.

For example :

(1) The event of getting 3 in a throw of a die is a simple event and the event of getting a multiple of 3 is a compound event as the event is composed of two simple events namely 3 or 6.

(2) In a toss of two coins together the event of getting a head is a compound event as it is composed of two simple events namely HT or TH.

Mutually Exclusive Events :

Two a more events in a trial on said to be mutually exclusive if the occurrence of one of the events prevents the occurrence of the other events.

For example (1) in tossing of a coin head and tail are mutually exclusive events as when head occurs tail cannot occur and vice-versa.

(2) In a draw of a card from a pack of cards, getting an ace and king and mutually exclusive since when ace occurs king cannot occur and vice-versa.

Equally Likely events or Cases :

Two or more events in a trial are said to be equally likely events if all the relevant evidences associated with the random experiment ensures that no event gets the preference of occurrence over the other events.

For example

(1) in a toss of a coin head and tail are equally likely events provided the coin is free from any defect.

(2) If unbiasedness of a die can be ensured then all the six events 1, 2, 3, 4, 5 and 6 associated with the throwing of a die will be equally like by events.

(3) To ensure all the 52 outcomes associated with the drawing of a card from a pack of 52 cards to be equally likely cards must be well shuffled before the draw in made.

Exhaustive Events or Cases :

The total number of possible outcomes associated with a random experiment is known as the exhaustive events a cases.

For example :

(1) in tossing of a coin, then are two exhaustive events or cases namely head and tail.

(2) In tossing of a coin twice (or tossing of two coins together) there are $2^2 = 4$ exhaustive cases, namely HH, HT, TH, TT.

(3) Exhaustive cases in a toss of 3 coins together is given by $2^3 = 8$ namely HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

(4) In throwing of a die the exhaustive cases in given by 6 namely 1, 2, 3, 4, 5 and 6.

(5) When 2 dice are thrown together the exhaustive cases is $6^2 = 36$ and they are given by–

(1, 1),	(1, 2),	(1, 3),	(1, 4),	(1, 5),	(1, 6)
(2, 1),	(2, 2),	(2, 3),	(2, 4),	(2, 5),	(2, 6)
(3, 1),	(3, 2),	(3, 3),	(3, 4),	(3, 5),	(3, 6)
(4, 1),	(4, 2),	(4, 3),	(4, 4)	(4, 5),	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Favourable cases or Events :

The number of cases or events in a trial, the occurrence of which ensures the occurrence of a particular event an known as the favourable cases.

For example :

(1) In a toss of two coins together, the number of cases favourable to the occurrence of one head and one tail and two namely HT and TH.

(2) In a throw of a die, there are 3 cases namely 2, 4 and 6 which are favourable to the event of getting an even number.

(3) In a throw of 2 dice together the events (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2) ensures occurrence of the event of getting a total of 8 points. So the favourable cases in this case is 5.

(4) If a card is drawn at random from a well shuffled pack of 52 cards, then there are 4 aces which ensures the occurrence of the event of drawing an ace and hence the favourable number of cases. is 4.

Independent Events :

Two or more events associated with a random experiment are said to be independent if the occurrences or non-occurrence of one does not affect and is not affected by the occurrence of the other events.

For example, (1) In 3 successive tosses of an unbiased coin, the event of getting a head in the 3rd toss is independent of getting a head in the 1st and the 2nd toss.

(2) In two success draws of a card one after the other from a well shuffled park of 52 cards, the event of getting a king in the 2nd draw independent of the event of gitting a king in the 1st draw if the card drawn in the first draw is replaced before the 2nd draw is made.

Dependent Events :

Two or more events associated with a random experiment are soda to be dependent if the occurrence of one event affects the occurrence of the other events.

For example, in two successive draws of a card one after the other from a well-shuffled pack of 52 cards the event of getting a king in the 2nd draw is dependent on the event of getting a king in the 1st draw if the card drawn in the 1st draw is not replaced before the 2nd card is drawn.

Classical or Mathematical Approach to Probability :

Classical a mathematical definition of probability was given by James Bernonlli while attempting to obtain the quantitative measure of uncertainty.

Definition :

If a trial of a random experiment results in ‘ n ’ equally likely mutually exclusive and exhaustive cases and ‘ m ’ of them are favourable to the occurrence of an event ‘ A ’, then probability of occurrence of the event ‘ A ’ denoted by $P(A)$, is given by

$$P(A) = \frac{\text{Number of favourable cases}}{\text{Exhaustive cases}}$$

$$\Rightarrow P(A) = \frac{m}{n}$$

Since $m \geq 0$ and $n > 0$, so we have

$$P(A) \geq 0 \text{ (i)}$$

Again since $m \leq n$, so we have

$$P(A) \leq 1 \text{ (ii)}$$

Courbing (i) and (ii) we get

$$0 \leq P(A) \leq 1.$$

Note :

- (1) Probability of an event always lies between 0 and 1.

- (2) The ratio $m : (n - m)$ is interpreted as the odds in favour of A' and the ratio $(n - m) : m$ as 'the odds against A'.
- (3) The non-occurrence of the event 'A', denoted by \bar{A} is classed the complementary event of A and the happening number of cases favourable the happening of the event \bar{A} is $n - m$. Therefore, the probability of occurrence of \bar{A} , by definition is given by

$$\begin{aligned} P(\bar{A}) &= \frac{n - m}{n} \\ &= 1 - \frac{m}{n} \\ \Rightarrow P(\bar{A}) &= 1 - P(A) \\ \Rightarrow P(A) + P(\bar{A}) &= 1 \end{aligned}$$

that is the sum of the probabilities of occurrence and non-occurrence of the same event is always unity.

- (4) When $P(A) = 0$, the event A is called the impossible event as in this case $m = 0$. Again when $P(A) = 1$, the event 'A' is called the certain or sure event as in this case when $m = n$ the event A is certain to occur.

Example 1 : An unbiased coin is tossed twice. What is the probability of getting (i) a head and (ii) at least a tail?

Solution : Since two unbiased coins are tossed, the all possible outcomes are given as follows: HH, HT, TH, TT

\therefore Exhaustive number of cases is $n = 4$

(i) The number of cases favourable to the event of getting a head is $m = 2$, namely HT, and TH.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{2}{4} = \frac{1}{2}$$

(ii) The number of cases favourable to the event of getting at least a tail is $m = 3$ namely HT, TH and TT.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{3}{4}$$

Example 2 : If a perfect die is thrown at random, find the probability of getting a number (i) 3 (ii) in between 2 and 5 (iii) greater than 4 and (iv) multiple of 2.

Solution : Since a perfect die is thrown, the number on its upper face can be any one of 1, 2, 3, 4, 5 and 6.

\therefore Exhaustive number of cases is $n = 6$

(i) The number of cases favourable to event of getting 3 out of 6 cases is $m = 1$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{1}{6}$$

(ii) The numbers in between 2 and 5 are 3 and 4.

\therefore The number of cases favourable to the event of getting a number in between 2 and 5 that is the number is 3 or 4 is $m = 2$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$$

(iii) The number of cases favourable to the event of getting a number greater than 4 i.e. 5 or 6 is $m = 2$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$$

(iv) The number of cases favourable to the event of getting a number which is a multiple of 2 i.e. 2, 4 and 6 is $m = 3$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$$

Example 3 : In a throw of two dice together at random, find the probability that

- (i) Find die shows 1 (one).
- (ii) both the dice show the same number.
- (iii) the sum of the numbers on the dice is 7.
- (iv) the sum of the numbers on the dice is a multiple of 3.

Solution : A throw of two dice together a may yield any one of the following results :

(1, 1),	(1, 2),	(1, 3),	(1, 4),	(1, 5),	(1, 6)
(2, 1),	(2, 2),	(2, 3),	(2, 4),	(2, 5),	(2, 6)
(3, 1),	(3, 2),	(3, 3),	(3, 4),	(3, 5),	(3, 6)
(4, 1),	(4, 2),	(4, 3),	(4, 4)	(4, 5),	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Exhaustive number of cases is $n = 6^2 = 36$

(i) The number of cases favourable to the event where first die shows 1 is $m = 6$ namely (1, 1), (1, 2), (1, 3), (1, 4), (1, 5) and (1, 6).

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

(ii) Favourable cases to the event that both the dice show the same number are (1, 1) (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6) i.e.; $m = 6$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

(iii) Favourable cases to the event of getting sum of the two numbers is 7 are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2) and (6, 1), i.e.; $m = 6$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

(iv) Favourable cases to the event that the sum of the two numbers is a multiple of 3 are (1, 2), (1, 5) (2, 1) (2, 4) (3, 3) (3, 6) (4, 2) (4, 5) (5, 1), (5, 4) (6, 3) and (6, 6) i. e.; $m = 12$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{12}{36} = \frac{1}{3}$$

Example 4 : What is the probability that a leap year selected at random contains 53 Sundays?

Solution : A leap year consists of 366 days which contains 52 complete weeks that is 52 Sundays and 2 days extra. These 2 days may be any one of the following combinations :

- (i) Monday, Tuesday
- (ii) Tuesday, Wednesday
- (iii) Wednesday, Thursday
- (iv) Thursday, Friday
- (v) Friday, Saturday
- (vi) Saturday, Sunday
- (vii) Sunday, Monday

The leap year to have 53 Sundays, one day of the extra 2 days must be a Sunday.

We see that of the above 7 equally likely cases only cases (vi) and (vii) are in favour of the 53 of Sundays.

$$\therefore \text{Required probability} = \frac{2}{7}$$

Example 5 : A bag contains 5 white and 3 black balls. A ball is drawn at random from the bag what is the probability that it is a white ball?

Solution : The bag contains altogether $5 + 3 = 8$ balls. A ball can be drawn at random from 8 balls in ${}^8C_1 = 8$ ways.

\therefore Exhaustive cases = $n = 8$

The number of cases favourable to the event of drawing a white ball from 5 white balls is ${}^5C_1 = 5$ ways ie, $m = 5$.

\therefore Required probability = $\frac{m}{n} = \frac{5}{8}$

Example : A person draws 3 balls at random from a bag containing 6 red and 5 black balls.

Find the probability that balls drawn are

- (i) all red
- (ii) 2 red and 1 black
- (iii) 1 red and 2 black

Solution : Total number of balls in the bag = $6 + 5 = 11$

Since 3 balls can be drawn from 11 balls in ${}^{11}C_3$ ways. So the exhaustive number of cases is given by $n = {}^{11}C_3$

$$= \frac{\angle 11}{\angle 3 \angle 8} = 165$$

(i) Since all the 3 balls drawn are required to be red, So 3 red balls out of 6 red balls can be drawn 6C_3 ways.

\therefore The favourable number of cases to the event of occurrence of 3rd balls is

$$m = {}^6C_3$$

$$= \frac{\angle 6}{\angle 3 \angle 3} = 20$$

\therefore Required Probability = $\frac{m}{n} = \frac{20}{165}$

(ii) In this case of the 3 balls drawn 2 balls are required to be red and one black ball. Now 2 red balls can be drawn out of 6 red in 6C_2 ways and 1 black ball can be drawn

out of 5 black in 5C_1 ways. Therefore, the total number of ways of drawing 2 red and 1 black ball is ${}^6C_2 \times {}^5C_1$

\therefore The number of cases favourable to the event of drawing 2 red and 1 black ball is

$$m = {}^6C_2 \times {}^5C_1 = 15$$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{15}{165}$$

(iii) In this case the event is to draw 1 red and 2 black balls.

Now, 1 red out of 6 red balls can be drawn 6C_1 ways and 2 black balls can be drawn in 5C_2 ways. So, the total number of ways of drawing 1 red and 2 black balls is given by ${}^6C_1 \times {}^5C_2$.

$$\therefore \text{The favourable cases} = m = {}^6C_1 \times {}^5C_2 = 10$$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{10}{165}$$

Example 7 : If a card is drawn at random from a well-shuffled pack of 52 cards, find the probability that the card drawn is

- (i) an ace
- (ii) a spade
- (iii) not a spade
- (iv) a diamond or a spade
- (v) a king or a queen

Solution : A card can be drawn from a pack of 52 well-shuffled cards in ${}^{52}C_1$ ways.

$$\therefore \text{Exhaustive cases} = n = {}^{52}C_1 = 52$$

(i) There are 4 aces in a pack of cards and an ace can be drawn from 4 in 4C_1 ways.

$$\therefore \text{Favourable cases} = m = {}^4C_1 = 4$$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

(ii) There are 13 cards of spade in a pack of cards. One card can be drawn from 13 spade cards in ${}^{13}C_1$ ways.

\therefore Favourable cases to the event of drawing a spade card $m = {}^{13}C_1 = 13$.

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{13}{52} = \frac{1}{4}$$

(iii) Probability that the card drawn is not a spade is

= 1 – probability that the card drawn is a spade card.

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Or

The number of cases favourable to the event of drawing a card which is not a

spade card = $m = {}^{39}C_1 = 39$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{39}{52} = \frac{3}{4}$$

(iv) Each of diamond and spade contains 13 cards in a pack. A card of each suit can be drawn in ${}^{13}C_1$ ways.

\therefore The number of cases favourable to the event of drawing a card of diamond or a spade

$$\begin{aligned} &= m = {}^{13}C_1 + {}^{13}C_1 \\ &= 13 + 13 = 26 \end{aligned}$$

$$\begin{aligned} \therefore \text{Required probability} &= \frac{m}{n} \\ &= \frac{26}{52} \\ &= \frac{1}{2} \end{aligned}$$

(v) There are altogether 8 cards of king and queen in a pack and each of king and queen has 4 cards. A card of each of king and queen can be drawn in 4C_1 ways.

\therefore The number of cases favourable to the event of drawing a king or a queen

$$\begin{aligned} &= m = {}^4C_1 + {}^4C_1 \\ &= 4 + 4 = 8 \end{aligned}$$

$$\therefore \text{Required probability} = \frac{m}{n} = \frac{8}{52} = \frac{2}{13}$$

Example 8 : An urn contains 20 balls marked with numbers from 1 to 20. If a ball is drawn at random from the urn, find the probability that the number on the ball is a multiple of (i) 2 and 5 and (ii) 3 and 5.

Solution : One ball can be drawn out of 20 balls in ${}^{20}C_1$ ways.

\therefore Exhaustive cases = $n = {}^{20}C_1 = 20$

(i) The multiples of 2 upto 20 are 2, 4, 6, 8, 10, 12, 14, 16, 18 and 20 and the multiples of 5 upto 20 are 5, 10, 15, and 20.

So the number of cases favourable to the event of getting a ball with number which is a (a) multiple 2 is 10 and (b) multiple of 5, is 4.

Since the two numbers 10 and 20 are multiples of both 2 and 5, so the distinct number of favourable cases to the event that the number on the ball is a multiple 2 or 5 is $m = 10 + 4 - 2 = 14$

\therefore Required probability $\frac{m}{n} = \frac{14}{20} = \frac{7}{10}$

(ii) The multiples of 3 upto 20 are

3, 6, 9, 12, 15 and 18

and the multiples of 5 upto to are

5, 10, 15, and 20

So the number of favourable cases to the event that the number on the ball is a multiple of 3 is 6 and the event that the number on the ball is a multiple of 5 is 4. Since the number 15 is a multiple of both 3 and 5. So the distinct number of cases favourable to the event of getting a number on the ball which is a multiple of 3 or 5 is $m = 6 + 4 - 1 = 9$

\therefore Required probability = $\frac{m}{n} = \frac{9}{20}$

Limitations of the Classical Approach to probability :

Classical definition of probability is based on two assumptions namely (i) the all possible outcomes of the random experiment are equally likely and (ii) the exhaustive cases i.e., the total number of possible outcomes is finite or unknown. The classical approach breaks down if any one of the two assumptions does not hold.

Following are a few examples where classical approach fails :

1. What is the probability that a candidate appearing in a certain last will pass?

Since the two possible outcomes viz; success and failure are not equally likely. So, the probability of passing the test by the candidate is certainly not $\frac{1}{2}$ or 50%.

2. What is the probability that a man jumps from a three storied building will survive?

Since in this case the two possible outcomes name by survival and death associated with the trial are not equally likely, So the probability of survival of the man of the is not $\frac{1}{2}$ or 50%.

3. What is the probability of getting 60 heads in a repeated toss of a perfect coin?

In this case we can not find the probability of occurrence of 50 hands, since the total number of tosses required to hour 50 heads is not known.

In the following different terminologies used in classical approach an redefined using set theory :

Sample space : The set of all possible outcomes associated with a given random experiment is called a sample space. It is also known as the outcome space of the random experiment. Each possible outcome of the experiment which is also the element of the sample space is called a sample point. Sample space is usually denoted by 'S'.

For example :

(1) in a random toss of 3 perfect coins together the sample space containing all possible outcomes is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(2) In a throw of an unbiased die the sample space of all possible outcomes is given by

$$S = \{1, 2, 3, 4, 5, 6\}$$

(3) If two unbiased dice are thrown together the sample space consisting of all possible out comes in given by $S = \{1, 2, 3, 4, 5, 6\}$

$$S = \{(1, 1), (1, 2), \text{---}, (1, 6), (2, 1) \text{---} (2, 6) \text{---} \\ \text{---} (6, 1), \text{---} (6, 6)\}$$

The number of sample points in the sample space 'S' is denoted by $n(S)$ which represents the exhaustive number of cases associated with the random experiment. In the above examples $n(S) = 8$ in (1) $n(S) = 6$ in (2) and $n(S) = 36$ in (3).

Remark :

1. A sample space is said to be finite/infinite if the number of elements in S is finite/infinite.
2. A sample space is called discrete if it contains only countably finite a infinite number of points which can be arranged into a countable sequence like $w_1, w_2 \text{---}$. While a sample space containing uncountably infinite number of points is called a continuous sample space.

Event :

Every non-empty subset A of the sample space is called an event.

Simple or Elementary event :

An elementary or simple event A is a non-empty subset consisting of single element of the sample space S.

For example, the event of getting '1' on both the upper faces of two dice thrown simultaneously given by the subset $A = \{(1, 1)\}$, is a simple event.

Compound or Composite Event :

A compound or composite event A is a sub-set of elements of the sample space S containing more than one element or elementary events of S.

For example, in a throw of an unbiased die, the sample space is given by

$$S = \{1, 2, 3, 4, 5, 6\}$$

Now, the event of getting an even number is a compound event given by the sub-set

$$A = \{2, 4, 6\}$$

In this case the event 'A' can be decomposed into a number of Simple events as follows :

$$A_1 = \{2\}, A_2 = \{4\}, A_3 = \{6\}$$

Remark : The null set ϕ represents the impossible event and the sample space S represents the certain or sure event.

For example, in a throw of two unbiased dice, the event A_1 of getting a sum of the points on the upper faces greater than 12 is $A_1 = \{\phi\}$, So, A_1 is an impossible event. Again the event A_2 of getting a sum of points greater than or equal to 2 and less than or equal to 12 is a certain or sure event since $A_2 = S$.

Mutually Exhaustive Events : Two events A and B associated with a given random experiment are said to be mutually exclusive if

$$A \cap B = \phi$$

For the 3 events A, B and C to be mutually exclusive we must have

$$A \cap B \cap C = \phi$$

That is for the events to be mutually exclusive, they must be disjoint events.

For example, in a toss of an unbiased coin at random three times, let us consider the following events :

A be the event of getting three heads.

B be the event of getting exactly one head.

C be the event of getting three tails.

Then the Sample space S , and the events A , B and C are given by the sets :

$$S = \{HHH, HHT, HTH, THH, HTT, THT, THT, TTT\}.$$

$$A = \{HHH\}, B = \{HTT, THT, TTH\} \text{ and } C = \{TTT\}.$$

Now find that

$$A \cap B = \phi$$

$$A \cap C = \phi$$

$$A \cap C = \phi$$

and $A \cap B \cap C = \phi$

Therefore, A , B and C are mutually exclusive events.

In a random throw of a dice let A denote the event of getting an even number and B denote the event of getting a number which is a multiple of 3, then the sample space S and the events A and B are given by

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2, 4, 6\}$$

and $B = \{3, 6\}$

We find that

$$A \cap B = \{6\} \neq \phi$$

Therefore, A and B are not mutually exclusive events.

EXERCISE

1. Fill in the blanks :

- (i) Theory of probability had its origin in _____.
- (ii) Two events are said to be mutually exclusion (disjoint) when they _____ simultaneously.
- (iii) Probability is a _____ increase of _____.
- (iv) Probability lies between _____ and _____.
- (v) ${}^{10}C_2 = \text{_____}$
- (vi) ${}^{12}C_3 = \text{_____}$
- (vii) The probability that the throw of two dice yields a total of 11 is _____.
- (viii) If a coin is tossed together, the probability of getting either a head or a tail is _____.

- (ix) If $P(A) = 0$, then A is ——— event.
 If $p(A)=1$, then A is ——— event.

[Ans. : (i) Gambling (ii) can not happen
 (iii) quantitative, uncertainty (iv) 0, 1 (v) 45 (vi) 110
 (vii) $\frac{2}{36}$ (viii) 1 (ix) an impossible (x) certain]

2. State True (T) or False (F) :

- (i) Mutually exclusive events and those in which events can occur simultaneously.
 (ii) Independent events are those in which the occurrence of one does not affect and is not affected by the other.
 (iii) The probability of getting 2 or 5 is $\frac{2}{36}$.
 (iv) The classical approach to probability is the oldest.
 (v) In a throw of a die the probability of getting 8 is $\frac{1}{6}$.
 (vi) If two events are not mutually exclusive they must be independent events.
 (vii) Classical approach to probability fails if the exhaustive cases in known.
 (viii) Equally likely of events is a condition for measure of probability is classical approach.
 (ix) If an event A can never take place, then $P(A) = 0$.

[Ans. : (i) F, (ii) T (iii) – F (iv) – T (v) – F
 (vi) T (vii) – F (viii) T (ix) T]

3. Write in short the concept of probability.

4. Define random experiment with examples.

5. Explain the terms with examples :

- (i) Trial (ii) Mutually exclusive events (iii) Equally likely events (iv) Exhaustive cases.

6. (a) State the mathematical definitions of probability.

(b) How that probability always his between 0 and 1.

(c) Interpret the values of $P(A) = 0$ and $P(A) = 1$.

7. (a) Mention the different approaches to probability.

(b) Define probability using any one of the approaches.

(c) Write the limitations of classical approach.

(d) Give at least one example where probability can not be measured using classical approach.

8. Explain the terms mutually exclusive and independent events. Give one example for each.
9. Distinguish between the following with examples :
- Simple and compound events.
 - Independent and dependent events.
 - Mutually exclusive and independent events.
10. Three perfect coins are tossed at random, find the probability of occurrences of (i) one head (ii) at least 2 heads.

$$[\text{Ans. : (i) } \frac{3}{8}, \text{ (ii) } \frac{1}{2}]$$

11. What is the probability that a non-leap year selected at random has 53 Sundays?

$$[\text{Ans. : } \frac{1}{7}]$$

12. If two unbiased dice can thrown simultaneously, what is the probability of getting
- at least a 6
 - a total of 9 points
 - a total which is a multiple of 4?

$$[\text{Ans. : (i) } \frac{11}{36} \text{ (ii) } \frac{1}{9} \text{ (iii) } \frac{1}{4}]$$

13. A bag contains 5 white, 6 red and 4 black balls. Three balls are drawn at random from the bag. Find the probability of drawing
- the balls of different colours
 - 3 red balls
 - 2 white and 1 black ball
 - 2 red and 1 white ball.
 - either 3 white or 3 black balls.

$$[\text{Ans. : (i) } \frac{120}{455} \text{ (ii) } \frac{20}{455} \text{ (iii) } \frac{40}{455} \text{ (iv) } \frac{15}{455} \text{ (v) } \frac{{}^5C_3 + {}^4C_3}{455} = \frac{14}{455}]$$

14. What is the probability of getting a number (i) less than 4 and (ii) multiple of 3 in a single throw of an unbiased die.

$$[\text{Ans. : (i) } \frac{1}{2} \text{ (ii) } \frac{1}{3}]$$

15. If two cards are drawn at random from a pack of 52 cards, find the probability of drawing

(i) two aces

(ii) an ace and a queen

(iii) a spade and a hearts

(iv) either 2 aces or 2 spade.

(v) two cards of different coloums :

$$[\text{Ans. : (i) } \frac{6}{1326} \text{ (ii) } \frac{16}{1326} \text{ (iii) } \frac{169}{1326} \text{ (iv) } \frac{{}^4C_2 + {}^{13}C_2}{1326} = \frac{78}{1326}$$

$$\text{(v) } \frac{{}^{26}C_1 \times {}^{26}C_1}{1326} = \frac{676}{1326}]$$
