INDIAN FOREST SERVICE (MAIN) EXAM-2022

JHGT-U-MTH

MATHEMATICS

Paper – I

Time Allowed : ***Three*** *Hours*

Maximum Marks : ***200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are ***EIGHT*** *questions in all, out of which* ***FIVE*** *are to be attempted.*

Questions no. ***1*** *and* ***5*** *are* ***compulsory****. Out of the remaining* ***SIX*** *questions,* ***THREE*** *are to be attempted selecting at least* ***ONE*** *question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

Answers must be written in ***ENGLISH*** *only.*

SECTION A

Q1. (a) Let U and W be subspaces of a vector space V and $x, y \in V$. Then prove that $x + U \subseteq y + W$ iff $U \subseteq W$ and $x - y \in W$. *8*

(b) Let $v_1 = (1, 1, -1)$, $v_2 = (4, 1, 1)$, $v_3 = (1, -1, 2)$ be a basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that $Tv_1 = (1, 0)$, $Tv_2 = (0, 1)$, $Tv_3 = (1, 1)$. Describe the linear transformation T. *8*

(c) Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$. *8*

(d) If $x + y + z = u$, $y + z = uv$, $z = uvw$, then determine $\frac{\partial(x, y, z)}{\partial(u, v, w)}$. *8*

(e) A variable plane is at a constant distance of 6 units from the origin and meets the axes in A : (a, 0, 0), B : (0, b, 0) and C : (0, 0, c). Find the locus of the centroid of the triangle ABC. *8*

Q2. (a) Are the matrices $A = \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ similar ? Justify your answer. *10*

(b) Using Lagrange's undetermined multipliers method, find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. *15*

(c) Obtain the coordinates of the points where the shortest distance line between the straight lines $\frac{x-3}{-1} = \frac{y-2}{2} = \frac{z-2}{-1}$; $\frac{x-2}{2} = \frac{y+3}{3} = \frac{z+2}{2}$ meets them. Also find the magnitude of the shortest distance and the equation of the shortest distance line between the straight lines mentioned above *15*

Q3. (a) Reduce the following quadratic form over the real field $\mathbb{R}$ to orthogonal form : *10*

$$q(x, y, z) = x^2 + 5y^2 - 4z^2 + 2xy - 4xz$$

(b) Find the centre of mass of a solid bounded below by $x^2 + y^2 \leq 4$, $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$. Take the density of the solid as uniform. *15*

(c) Find the equation of the cylinder whose generators intersect the curve, $2x^2 + 3y^2 = 4z$, $x - y + 2z = 3$ and are parallel to the line $3x = -2y = 4z$. *15*

Q4. (a) Show that the improper integral

$$\int_0^\infty x^{m-1} e^{-x}\, dx$$

is convergent if $m > 0$. *10*

(b) Let V be the complex vector space of 3×3 skew-symmetric matrices with complex entries i.e. $V = \{A \in M_{3 \times 3}(\mathbb{C}) \mid A^t = -A\}$.

$$\text{Let } B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Define a linear transformation $T : V \to V$ by $T(A) = BA - AB$. Find the eigenvalues and eigenvectors of T. *15*

(c) Reduce the equation,

$$3x^2 + 6yz - y^2 - z^2 - 6x + 6y + 2z + 2 = 0$$

to a canonical form and mention the name of the surface it represents. *15*

SECTION B

Q5. (a) Find the general solution of the differential equation :

$$p^2 \cos^2 y + p \sin x \cos x \cos y - \sin y \cos^2 x = 0, \text{ where } p \equiv \frac{dy}{dx}.$$ *8*

(b) Solve the differential equation :

$$(D^2 - 1)y = e^x (1 + x^2), \text{ where } D \equiv \frac{d}{dx}.$$ *8*

(c) Three forces P, Q and R act along the sides BC, CA and AB of Δ ABC in order to keep the system in equilibrium. If the resultant force touches the inscribed circle, then prove that

$$\frac{1 + \cos \alpha}{P} + \frac{1 + \cos \beta}{Q} + \frac{1 + \cos \gamma}{R} = 0,$$

where α, β, γ are the interior angles subtended at A, B, C respectively. *8*

(d) A person is drawing water from a well with a light bucket which leaks uniformly. The bucket weighs 50 kg when it is full. When it arrives at the top, half of the water remains inside. If the depth of the water level in the well from the top is 30 m, then find the work done in raising the bucket to the top from the water level. *8*

(e) Determine constants a, b, c so that the directional derivative of $\phi(x, y, z) = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 88 in a direction parallel to z-axis. *8*

Q6. (a) Solve the differential equation :

$$4(xp^2 + yp) = y^4, \text{ where } p \equiv \frac{dy}{dx}.$$ *10*

(b) A particle of mass m, which is attached to one end of a light string whose other end is fixed at a point O, describes a circular motion in a horizontal plane about the vertical axis through O. Prove that the particle moves in a conical pendulum only if $g < l\omega^2$, where l is the length of the string and ω being angular velocity.

Further, a particle of mass m is attached to the middle of a light string of length $2l$, one end of which is fastened to a fixed point and the other end to a smooth ring of mass M which slides on a smooth vertical rod. If the particle describes a horizontal circle with uniform angular velocity ω about the rod, then prove that the inclination of both portions of the string to the vertical is

$$\cos^{-1}\left\{\frac{(m + 2M)g}{m\, l\, \omega^2}\right\}.$$ *15*

(c) Given that C is a curve of the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 2$ and C is described counterclockwise.

Verify Stokes' theorem for the line integral $\int_C - y^3 dx + x^3 dy - z^3 dz$. *15*

Q7. (a) Derive vector identity for divergence of cross product of two vector point functions. Given a relation between linear and angular velocity as $\vec{v} = \vec{\omega} \times \vec{r}$.

If $\vec{\omega}$ is constant, then show that

(i) curl $\vec{v} = 2\vec{\omega}$

(ii) div $\vec{v} = 0$. *10*

(b) Given that $y_1 = x^2$ is a solution of the differential equation,

$$x^3 \frac{d^2y}{dx^2} - (x^2 + 3x)\frac{dy}{dx} + 6y = 0,$$

find the other linearly independent solution of the above differential equation and write down the general solution of the differential equation. *15*

(c) PR and QR are two equal heavy strings tied together at R and carrying a weight W at R. P and Q are two points in the same horizontal line and 2a is the distance between them. l is the length of each string and h is the depth of R below PQ. Prove that

(i) $l^2 - h^2 = 2c^2 \left(\cosh \frac{a}{c} - 1\right)$,

(ii) Tension at P or Q = $\frac{1}{2h}\{lW + (l^2 + h^2)w\}$,

where c is the parameter of the catenary and w is the line density of the string. *15*

Q8. (a) A bucket is in the form of a frustum of a cone and is filled with water of density ρ. If the bottom and top ends of the bucket have radii a and b respectively and h is the height of the bucket, then find the resultant vertical thrust on the curved surface of the bucket. Is that thrust equal to $\frac{1}{3}\pi \rho g h (b - a)(b + 2a)$? *10*

(b) Solve the differential equation :

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 15x^4 + 8x^3.$$ *15*

(c) If a curve in a space is represented by $\vec{r} = \vec{r}(t)$, then derive expressions of its torsion and curvature in terms of $\dot{\vec{r}}$, $\ddot{\vec{r}}$ and $\dddot{\vec{r}}$. Find the curvature and torsion of the curve given by $\vec{r} = (at - a \sin t,\ a - a \cos t,\ bt)$. *15*